

# On $v$ -adic periods of $t$ -motives

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## Abstract

In this paper, we prove the equality between the transcendental degree of the field generated by the  $v$ -adic periods of a  $t$ -motive  $M$  and the dimension of the Tannakian Galois group for  $M$ , where  $v$  is a “finite” place of the rational function field over a finite field. As an application, we prove the algebraic independence of certain “formal” polylogarithms.

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# 1 Introduction

Let  $\mathbb{F}_q$  be the finite field with  $q$  elements,  $\theta$  and  $t$  be variables independent from each other, and  $v \in \mathbb{F}_q[t]$  a fixed monic irreducible polynomial of degree  $d$ . Let  $M$  be a rigid analytically trivial  $t$ -motive over  $\overline{\mathbb{F}_q(\theta)}$ . Then there exists an  $\infty$ -adic period matrix for the Betti realization of  $M$ . Set  $\Lambda$  to be the field generated by the components of this matrix over  $\overline{\mathbb{F}_q(\theta)}(t)$ . Set  $\Gamma$  to be the Tannakian Galois group of  $M$  with respect to the Betti realization. Papanikolas [10] shows that the transcendental degree of  $\Lambda$  over  $\overline{\mathbb{F}_q(\theta)}(t)$  coincides with the dimension of  $\Gamma$ . In this paper, we prove the  $v$ -adic analogue of this theorem.

Let  $K/\mathbb{F}_q$  be a regular extension of fields. We set  $K^{\text{sep}}[t]_v := \varprojlim (K^{\text{sep}}[t]/v^n)$  and  $K^{\text{sep}}(t)_v := \mathbb{F}_q(t) \otimes_{\mathbb{F}_q[t]} K^{\text{sep}}[t]_v$ , where  $K^{\text{sep}}$  is a separable closure of  $K$ . We also define  $\mathbb{F}_q(t)_v$  and  $K(t)_v$  by the same way. Let  $\sigma$  be the ring endomorphism  $\sum a_i t^i \mapsto \sum a_i^q t^i$  of  $K^{\text{sep}}[t]$ . Then  $\sigma$  naturally extends to an endomorphism of  $K^{\text{sep}}(t)_v$ , also denoted by  $\sigma$ . A  $\varphi$ -module over  $K(t)_v$  is a pair  $(M, \varphi)$  (or simply  $M$ ) where  $M$  is a  $K(t)_v$ -vector space and  $\varphi : M \rightarrow M$  is an additive map such that  $\varphi(ax) = \sigma(a)\varphi(x)$  for all  $a \in K(t)_v$  and  $x \in M$ . A morphism of  $\varphi$ -modules is a  $K(t)_v$ -linear map which is compatible with the  $\varphi$ 's. A tensor product of two  $\varphi$ -modules is defined naturally.

For any  $\varphi$ -module  $M$ , we define the  $v$ -adic realization of  $M$ :

$$V(M) := (K^{\text{sep}}(t)_v \otimes_{K(t)_v} M)^\varphi,$$

where  $\varphi$  acts on  $K^{\text{sep}}(t)_v \otimes_{K(t)_v} M$  by  $\sigma \otimes \varphi$  and  $(-)^{\varphi}$  is the  $\varphi$ -fixed part. Then there exists a natural map

$$\iota_M : K^{\text{sep}}(t)_v \otimes_{\mathbb{F}_q(t)_v} V(M) \rightarrow K^{\text{sep}}(t)_v \otimes_{K(t)_v} M.$$

We can prove that  $\iota_M$  is injective for each  $\varphi$ -module  $M$ . A  $\varphi$ -module  $M$  is said to be  $K^{\text{sep}}(t)_v$ -trivial if  $M$  is finite-dimensional over  $K(t)_v$  and  $\iota_M$  is an isomorphism. Then the category of  $K^{\text{sep}}(t)_v$ -trivial  $\varphi$ -modules over  $K(t)_v$  equipped with the functor  $V$  forms a neutral Tannakian category over  $\mathbb{F}_q(t)_v$ . For any  $K^{\text{sep}}(t)_v$ -trivial  $\varphi$ -module  $M$ , we denote by  $\Gamma_M$  the Tannakian Galois group of the Tannakian subcategory of  $K^{\text{sep}}(t)_v$ -trivial  $\varphi$ -modules generated by  $M$  (see Subsections 3.2 and 3.3).

Let  $M$  be a finite-dimensional  $\varphi$ -module and  $\mathbf{m} \in \text{Mat}_{r \times 1}(M)$  a  $K(t)_v$ -basis of  $M$ . Then there exists a matrix  $\Phi \in \text{Mat}_{r \times r}(K(t)_v)$  such that  $\varphi \mathbf{m} = \Phi \mathbf{m}$ . If  $M$  is  $K^{\text{sep}}(t)_v$ -trivial, we can take a matrix  $\Psi = (\Psi_{ij})_{i,j} \in \text{GL}_r(K^{\text{sep}}(t)_v)$  such that  $\Psi^{-1} \mathbf{m}$  forms an  $\mathbb{F}_q(t)_v$ -basis of  $V(M)$ . The entries of this matrix are called  $v$ -adic periods of  $M$ , which are our main objects of study in this paper. We set

$$\Sigma := K(t)_v[\Psi, 1/\det \Psi] := K(t)_v[\Psi_{11}, \Psi_{12}, \dots, \Psi_{rr}, 1/\det \Psi] \subset K^{\text{sep}}(t)_v.$$

Then  $\Sigma$  is stable under the  $\sigma$ -action. For any  $\mathbb{F}_q(t)_v$ -algebra  $R$  and  $S$ , we set  $S^{(R)} := R \otimes_{\mathbb{F}_q(t)_v} S$ . If  $\sigma$  acts on  $S$ , we define the  $\sigma$ -action on  $S^{(R)}$  by  $\text{id} \otimes \sigma$ . Set  $\Gamma(R) := \text{Aut}_\sigma(\Sigma^{(R)}/K(t)_v^{(R)})$  the group of automorphisms of  $\Sigma^{(R)}$  over  $K(t)_v^{(R)}$  that commute with  $\sigma$ . Then  $\Gamma$  forms a functor from the category of  $\mathbb{F}_q(t)_v$ -algebras to the category of groups. If we factorize  $v = \prod_{l \in \mathbb{Z}/d} (t - \lambda_l)$  in  $K^{\text{sep}}[t]$  with  $\lambda_l^q = \lambda_{l+1}$ , then we can write  $K^{\text{sep}}(t)_v = \prod_l K^{\text{sep}}((t - \lambda_l))$  and  $\Psi_{ij} = (\Psi_{ijl})_l$  where  $\Psi_{ijl} \in K^{\text{sep}}((t - \lambda_l))$ . We set

$$\Lambda_l := K(t)_v(\Psi_{11l}, \dots, \Psi_{rrl}) \subset K^{\text{sep}}((t - \lambda_l))$$

for each  $l \in \mathbb{Z}/d$ . Our main result in this paper is (see Lemma 4.16 and Theorems 4.14 and 5.15):

**Theorem 1.1.** *The functor  $\Gamma$  is representable by a smooth affine algebraic variety over  $\mathbb{F}_q(t)_v$ , also denoted by  $\Gamma$ . We have an equality  $\dim \Gamma = \text{tr.deg}_{K(t)_v} \Lambda_l$  for each  $l \in \mathbb{Z}/d$  and there exists a natural isomorphism  $\Gamma \rightarrow \Gamma_M$  of affine group schemes over  $\mathbb{F}_q(t)_v$ .*

This theorem is a  $v$ -adic analogue of Papanikolas's Theorem 4.3.1 and 4.5.10 in [10], which treated  $\infty$ -adic objects. The proof of this theorem follows [10] closely, but since  $K^{\text{sep}}(t)_v$  is not a field if  $d > 1$ , several arguments here are more complicated than in [10]. Let  $K = \mathbb{F}_q(\theta)$  where  $\theta$  is a variable independent of  $t$ . Papanikolas shows the equality of the transcendental degree of the field of periods (specialized at  $t = \theta$ ) over  $K$  and the dimension of the Tannakian Galois group using the so-called ABP-criterion proved by Anderson, Brownawell and Papanikolas in [2]. In fact he proved an algebraic independence of Carlitz logarithms. On the other hand, Anderson and Thakur [3] shows that the relation between the Carlitz zeta values and Carlitz logarithms. Then using these results, Chang and Yu [5] determined the all algebraic relations among the Carlitz zeta values. These applications are our motivation of this paper, but in this paper, we can only prove a  $v$ -adic analogue of the ABP-criterion for the rank one case.

In Section 3, first we review a theory of  $\varphi$ -modules in a general setting and construct a Tannakian category. In the  $v$ -adic case, we show that this category is equivalent to the category of Galois representations. In Section 4, we consider Frobenius equations in our situation, and construct  $\Gamma$ . In Section 5, we discuss the relation between  $\Gamma$  and  $\Gamma_M$ , and prove that these are isomorphic in the  $v$ -adic case. This uses the fact that the set of  $\mathbb{F}_q(t)_v$ -valued points  $\Gamma(\mathbb{F}_q(t)_v)$  is Zariski dense in  $\Gamma$ . Since  $\Gamma(\mathbb{F}_q(t)_v)$  contains the Galois image, this is large enough in  $\Gamma$ . This is an essentially different point from Papanikolas's proof for the  $\infty$ -adic case, in which the Zariski density is not proved and other facts are used to show this isomorphism. In Section 6, we discuss a  $v$ -adic analogue of the ABP-criterion. In Section 7, we prove the algebraic independence of certain “formal” polylogarithms.

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## 2 Notations and terminology

### 2.1 Table of symbols

$\mathbb{F}_q$	$:=$	the finite field of $q$ elements
$\bar{k}$	$:=$	an algebraic closure of a field $k$
$k^{\text{sep}}$	$:=$	the separable closure of a field $k$ in $\bar{k}$
$\#S$	$:=$	the cardinality of a set $S$
$\text{Mat}_{r \times s}(R)$	$:=$	the set of $r$ by $s$ matrices with entries in a ring or module $R$
$\text{GL}_r(R)$	$:=$	the group of invertible $r$ by $r$ matrices with entries in a ring $R$
$\text{Vec}(k)$	$:=$	the category of finite-dimensional vector spaces over a field $k$
$\text{Rep}(G, R)$	$:=$	for a ring $R$ the category of finitely generated $R$ -representations of an affine group scheme $G$ over $R$ , or for a topological ring $R$ the category of finitely generated continuous $R$ -representations of a topological group $G$

## 2.2 Action

Let  $R$  be a ring or module and  $f : R \rightarrow R$  a map. For a matrix  $A = (A_{ij})_{ij} \in \text{Mat}_{r \times s}(R)$ , we denote by  $f(A)$  the matrix  $(f(A_{ij}))_{ij}$ .

Let  $S$  be a set and  $H$  a set of maps from  $S$  to itself. Then we denote by  $S^H$  the subset of  $S$  consisting of elements which are fixed by  $H$ . For a map  $f : S \rightarrow S$ , we set  $S^f := S^{\{f\}}$ .

## 2.3 Base change

Let  $R \rightarrow S$  be a homomorphism of commutative rings and  $X$  a scheme over  $R$ . We denote by  $X_S := X \times_{\text{Spec } R} \text{Spec } S$  the base change from  $R$  to  $S$  of  $X$ . We also denote by  $X(S) := \text{Hom}_{\text{Spec } R}(\text{Spec } S, X)$  the set of  $S$ -valued points of  $X$  over  $R$ . When  $R$  and  $S$  are fields, we have a natural injection  $X(R) \hookrightarrow X(S) \cong X_S(S)$ . We always consider  $X(R)$  as a subset of  $X_S(S)$  via this injection.

## 3 $\varphi$ -modules

### 3.1 étale $\varphi$ -modules

In this subsection, we recall the definitions and properties of étale  $\varphi$ -modules (cf. [7]). Let  $A$  be a commutative ring and  $\sigma$  an endomorphism of  $A$ . For any  $A$ -module  $M$ , we put  $M^{(\sigma)} := A \otimes_A M$ , the scalar extension of  $M$  by  $\sigma$ . A map  $\varphi : M \rightarrow M$  is said to be  $\sigma$ -semilinear if  $\varphi(x + y) = \varphi(x) + \varphi(y)$  and  $\varphi(ax) = \sigma(a)\varphi(x)$  for all  $x, y \in M$  and  $a \in A$ . Then it is clear that to give a  $\sigma$ -semilinear map  $\varphi : M \rightarrow M$  is equivalent to giving an  $A$ -linear map  $\varphi_\sigma : M^{(\sigma)} \rightarrow M$ .

**Definition 3.1.** A  $\varphi$ -module  $(M, \varphi)$  over  $(A, \sigma)$  (or simply,  $M$  over  $A$ ) is an  $A$ -module  $M$  endowed with a  $\sigma$ -semilinear map  $\varphi : M \rightarrow M$ . A *morphism* of  $\varphi$ -modules is an  $A$ -linear map which is compatible with the  $\varphi$ 's. When  $A$  is a noetherian ring, a  $\varphi$ -module  $(M, \varphi)$  is said to be *étale* if  $M$  is a finitely generated  $A$ -module and  $\varphi_\sigma : M^{(\sigma)} \rightarrow M$  is bijective.

We denote by  $\Phi M_A$  the category of  $\varphi$ -modules over  $A$  and  $\Phi M_A^{\text{ét}}$  its full subcategory consisting of all étale  $\varphi$ -modules. For any  $\varphi$ -modules  $M$  and  $N$ , we denote by  $\text{Hom}_\varphi(M, N)$  the set of morphisms of  $M$  to  $N$  in  $\Phi M_A$ .

Let  $A_\sigma[\varphi]$  be the ring (non commutative if  $\sigma \neq \text{id}_A$ ) generated by  $A$  and an element  $\varphi$  with the relation

$$\varphi a = \sigma(a)\varphi$$

for each  $a \in A$ . Then it is clear that the category  $\Phi M_A$  and the category of  $A_\sigma[\varphi]$ -modules are naturally identified. Hence, the category  $\Phi M$  is an  $A^\sigma$ -linear abelian category.

For each  $\varphi$ -module  $M$  and  $N$ , we denote by  $M \otimes N$  the *tensor product* of  $M$  and  $N$ , which is  $M \otimes_A N$  as an  $A$ -module and has a  $\varphi$ -action defined by  $\varphi \otimes \varphi$ . Then the functor  $\otimes$  is a bi-additive functor and  $(A, \sigma)$  is an identity object in  $\Phi M_A$  for this tensor product. Therefore the category  $\Phi M_A$  is an abelian tensor category ([6]).

**Proposition 3.2.** *There exists a natural isomorphism  $A^\sigma \cong \text{End}_\varphi(A) := \text{Hom}_\varphi(A, A)$ .*

*Proof.* For any endomorphism  $f \in \text{End}_\varphi(A)$ , we have  $\sigma(f(1)) = \varphi(f(1)) = f(\varphi(1)) = f(\sigma(1)) = f(1)$ . Hence  $f(1) \in A^\sigma$ . Conversely for any element  $a \in A^\sigma$ , we have a map  $f_a : A \rightarrow A; x \mapsto ax$ . It is clear that  $f_a \in \text{End}_\varphi(A)$ . These are inverse to each other.  $\square$

**Proposition 3.3.** *Assume that  $A$  is noetherian and  $\sigma$  is flat. Then the category  $\Phi M_A^{\text{ét}}$  is an abelian  $A^\sigma$ -linear tensor category.*

*Proof.* It is clear that  $\Phi M_A^{\text{ét}}$  is closed under finite sums and tensor products, and the identity object  $(A, \sigma)$  is étale. Therefore it is enough to show that for each étale  $\varphi$ -modules  $M$  and  $N$  and a morphism  $f : M \rightarrow N$ , the kernel and cokernel of  $f$  in  $\Phi M_A$  are étale. Since  $M$  and  $N$  are étale and  $\sigma$  is flat, we have the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & (\ker f)^{(\sigma)} & \longrightarrow & M^{(\sigma)} & \longrightarrow & N^{(\sigma)} & \longrightarrow & (\text{im } f)^{(\sigma)} & \longrightarrow & 0 \\ & & \varphi'_\sigma \downarrow & & \varphi_{M,\sigma} \downarrow & & \varphi_{N,\sigma} \downarrow & & \varphi''_\sigma \downarrow & & \\ 0 & \longrightarrow & \ker f & \longrightarrow & M & \longrightarrow & N & \longrightarrow & \text{im } f & \longrightarrow & 0, \end{array}$$

where  $\varphi_{M,\sigma}$  and  $\varphi_{N,\sigma}$  are isomorphisms and the rows are exact. Then we have that  $\varphi'_\sigma$  and  $\varphi''_\sigma$  are isomorphism by a diagram chasing.  $\square$

Let  $(M, \varphi_M)$  and  $(N, \varphi_N)$  be  $\varphi$ -modules over  $A$ . If  $\varphi_{M,\sigma} : M^{(\sigma)} \rightarrow M$  is an isomorphism, we define a  $\varphi$ -module  $\text{Hom}(M, N)$ , whose underlying  $A$ -module is the space  $\text{Hom}_A(M, N)$  of  $A$ -module homomorphisms and a  $\varphi$ -action is defined by

$$\text{Hom}_A(M, N)^{(\sigma)} \rightarrow \text{Hom}_A(M^{(\sigma)}, N^{(\sigma)}) \rightarrow \text{Hom}_A(M, N),$$

where the first map is the natural map and the second map is defined by  $f \mapsto \varphi_{N,\sigma} \circ f \circ \varphi_{M,\sigma}^{-1}$ . There exists a natural morphism of  $\varphi$ -modules  $\text{ev}_{M,N} : \text{Hom}(M, N) \otimes M \rightarrow N$ . For each  $M$  such that  $\varphi_{M,\sigma}$  is an isomorphism, we set  $M^\vee := \text{Hom}(M, A)$  the dual of  $M$ .

**Proposition 3.4.** *Assume that  $A$  is noetherian and  $\sigma$  is flat. Then for any objects  $M$  and  $N$  in  $\Phi M_A^{\text{ét}}$ , the  $\varphi$ -module  $\text{Hom}(M, N)$  is étale, the contravariant functor*

$$\Phi M_A^{\text{ét}} \rightarrow \mathbf{Set}; \quad T \mapsto \text{Hom}_\varphi(T \otimes M, N)$$

*is representable by  $\text{Hom}(M, N)$  and  $\text{ev}_{M,N}$  corresponds to  $\text{id}_{\text{Hom}(M,N)}$ .*

*Proof.* Since  $M$  and  $N$  are finitely generated and  $A$  is noetherian,  $\text{Hom}(M, N)$  is also finitely generated. Since  $\sigma$  is flat and  $M$  is finitely presented, the map  $\text{Hom}_A(M, N)^{(\sigma)} \rightarrow \text{Hom}_A(M^{(\sigma)}, N^{(\sigma)})$  is an isomorphism ([4], Chap. I, Sect. 2, Prop. 11). Since  $\varphi_{M,\sigma}$  and  $\varphi_{N,\sigma}$  are bijective, the  $\varphi$ -module  $\text{Hom}(M, N)$  is étale. It is clear that there exists a natural isomorphism  $\text{Hom}_A(T \otimes M, N) \cong \text{Hom}_A(T, \text{Hom}(M, N))$  which is functorial in  $T$ . Then we can calculate that the subspaces  $\text{Hom}_\varphi(T \otimes M, N)$  and  $\text{Hom}_\varphi(T, \text{Hom}(M, N))$  are corresponding with this isomorphism. The last assertion is clear.  $\square$

**Proposition 3.5.** *Assume that  $A$  is a field. Then the category  $\Phi M_A^{\text{ét}}$  is a rigid abelian  $A^\sigma$ -linear tensor category.*

*Proof.* By Proposition 3.3,  $\Phi M_A^{\text{ét}}$  is an abelian  $A^\sigma$ -linear tensor category. By Proposition 3.4,  $\Phi M_A^{\text{ét}}$  has internal homs. Therefore it is enough to show that the natural map

$$\bigotimes_{i \in I} \text{Hom}(M_i, N_i) \rightarrow \text{Hom}(\bigotimes_{i \in I} M_i, \bigotimes_{i \in I} N_i)$$

is an isomorphism for any finite families of objects  $(M_i)_{i \in I}$  and  $(N_i)_{i \in I}$ , and the natural map

$$M \rightarrow M^{\vee\vee}$$

is an isomorphism for any object  $M$  ([6]). These are true because  $A$  is a field.  $\square$

### 3.2 $L$ -triviality

Let  $d$  be a positive integer and  $F \subset E \subset L$  ring extensions where  $F, E$  are fields and  $L = \prod_{l \in \mathbb{Z}/d} L_l$  is a finite product of fields. For each  $l$ , we sometimes consider  $L_l$  as a subset of  $L$  in an obvious way. Let  $\sigma : L \rightarrow L$  be a ring endomorphism. We assume that the triple  $(F, E, L)$  satisfies the following properties:

- $\sigma(E) \subset E$  and  $\sigma(L_l) \subset L_{l+1}$  for all  $l$ ,
- $F = E^\sigma = L^\sigma$ ,
- $L$  is a separable extension over  $E$ .

Such a triple  $(F, E, L)$  is called  $\sigma$ -admissible. See Lemma 3.24 for our main example. Another example can be found in [10].

Note that the separability of  $L$  over  $E$  is used to prove the smoothness of some algebraic groups (see Theorem 4.14), and not used in this section.

**Remark 3.6.** In [10], the term  $\sigma$ -admissible triple is defined only in the case where  $L$  is a field and  $\sigma$  is an isomorphism. Thus our general setting urges us to argue with greater care than in [10] at several points, and hence we decided not to avoid repeating similar arguments.

In this subsection, we consider  $\varphi$ -modules over  $(E, \sigma|_E)$ . For any  $\varphi$ -module  $M$  over  $E$ , we set

$$V(M) := (L \otimes_E M)^\varphi$$

where  $\varphi$  acts on  $L \otimes_E M$  by  $\sigma \otimes \varphi$ . Then  $V(M)$  is an  $F$ -vector space and  $V$  forms a functor. We have natural maps

$$\iota_M : L \otimes_F V(M) \rightarrow L \otimes_E M,$$

$$\iota_{M,l} : L_l \otimes_F V(M) \rightarrow L_l \otimes_E M \text{ for all } l.$$

**Lemma 3.7.** *Let  $M$  be a  $\varphi$ -module, and let  $\mu_1, \dots, \mu_m \in V(M)$ . If  $\mu_1, \dots, \mu_m$  are linearly independent over  $F$ , then they are linearly independent over  $L$  (in  $L \otimes_E M$ ).*

*Proof.* Assume that the lemma is not true. Then there exist  $m \geq 1$ ,  $\mu_1, \dots, \mu_m \in V(M)$  and  $f_1, \dots, f_m \in L$  such that,  $\mu_1, \dots, \mu_m$  are linearly independent over  $F$ ,  $(f_i)_i \neq 0$  and  $\sum_i f_i \mu_i = 0$ . We may assume that  $m$  is minimal among the integers which satisfy the above properties. We also assume that  $f_1 = (a_l)_l \in \prod_l L_l$  is non-zero. Let  $a_{l_0} \neq 0$ . Then there exists an element  $f' \in L$  such that  $f'f = e_{l_0}$ , where  $e_{l_0} \in L$  is the element such that the  $l_0$ -th component is one and the other components are zero. Then we have  $\sum_i f' f_i \mu_i = 0$ . Therefore we may assume that  $f_1 = e_{l_0}$ . Then we have

$$0 = \sum_{j=0}^{d-1} \varphi^j \left( \sum_{i=1}^m f_i \mu_i \right) = \sum_{i=1}^m \sum_{j=0}^{d-1} \varphi^j (f_i \mu_i) = \sum_{i=1}^m \left( \sum_{j=0}^{d-1} \sigma^j(f_i) \right) \mu_i = \mu_1 + \sum_{i=2}^m \left( \sum_{j=0}^{d-1} \sigma^j(f_i) \right) \mu_i.$$

Therefore we may assume that  $f_1 = 1$ . Then we have

$$0 = \varphi \left( \sum_{i=1}^m f_i \mu_i \right) - \sum_{i=1}^m f_i \mu_i = \sum_{i=1}^m (\sigma(f_i) - f_i) \mu_i = \sum_{i=2}^m (\sigma(f_i) - f_i) \mu_i.$$

By the minimality of  $m$ , we have  $f_i \in L^\sigma = F$  for all  $i$ . This contradicts the linear independence of  $(\mu_i)_i$  over  $F$ .  $\square$

**Corollary 3.8.** *For any  $\varphi$ -module  $M$ , the maps  $\iota_M$  and  $\iota_{M,l}$  are injective and we have  $\dim_F V(M) \leq \dim_E M$ .*

*Proof.* By Lemma 3.7,  $\iota_M$  is injective. It is clear that  $\iota_M$  is injective if and only if  $\iota_{M,l}$  are injective for all  $l$ . Therefore  $\iota_{M,l}$  is injective and we have an inequality  $\dim_F V(M) = \dim_{L_l}(L_l \otimes_F V(M)) \leq \dim_{L_l}(L_l \otimes_E M) = \dim_E M$ .  $\square$

**Definition 3.9.** Let  $M$  be a finite-dimensional  $\varphi$ -module over  $E$ . We say that  $M$  is  *$L$ -trivial* if the map  $\iota_M$  is an isomorphism.

We denote by  $\Phi M_E^L$  the full subcategory of  $\Phi M_E$  consisting of all  $L$ -trivial  $\varphi$ -modules. Let  $M$  be a finite-dimensional  $\varphi$ -module over  $E$  and  $\mathbf{m} \in \text{Mat}_{r \times 1}(M)$  its  $E$ -basis. Then there exists a matrix  $\Phi \in \text{Mat}_{r \times r}(E)$  such that  $\varphi \mathbf{m} = \Phi \mathbf{m}$ .

**Proposition 3.10.** *The following conditions are equivalent:*

- (1)  $M$  is  $L$ -trivial,
- (2)  $\iota_{M,l}$  is an isomorphism for each  $l$ ,
- (3)  $\iota_{M,l}$  is an isomorphism for some  $l$ ,
- (4)  $\dim_F V(M) = \dim_E M$ ,
- (5) there exists a matrix  $\Psi \in \text{GL}_r(L)$  such that  $\sigma \Psi = \Phi \Psi$ .

*Proof.* It is clear that (1)  $\iff$  (2)  $\Rightarrow$  (3)  $\Rightarrow$  (4). Assume that the condition (4) is true. Then for each  $l$ , we have  $\dim_{L_l}(L_l \otimes_F V(M)) = \dim_F V(M) = \dim_E M = \dim_{L_l}(L_l \otimes_E M)$ . Therefore  $\iota_{M,l}$  is an isomorphism. This means that the condition (4) implies the condition (2).

Assume that the condition (1) is true. Let  $\mathbf{x}$  be an  $F$ -basis of  $V(M)$ . Since the natural map  $\iota_M : L \otimes_F V(M) \rightarrow L \otimes_E M$  is an isomorphism, there exists a matrix  $\Psi \in \text{GL}_r(L)$  such that  $\Psi \mathbf{x} = 1 \otimes \mathbf{m}$ . Then we have

$$(\sigma \Psi) \mathbf{x} = (\sigma \Psi)(\varphi \mathbf{x}) = \varphi(\Psi \mathbf{x}) = \varphi(1 \otimes \mathbf{m}) = 1 \otimes \varphi \mathbf{m} = 1 \otimes \Phi \mathbf{m} = \Phi(1 \otimes \mathbf{m}) = \Phi \Psi \mathbf{x}.$$

By Lemma 3.7, we have  $\sigma \Psi = \Phi \Psi$  and the condition (5) is true. Conversely, assume that the condition (5) is true. Then we have

$$\varphi(\Psi^{-1}(1 \otimes \mathbf{m})) = (\sigma \Psi)^{-1}(1 \otimes \varphi \mathbf{m}) = (\Phi \Psi)^{-1}(1 \otimes \Phi \mathbf{m}) = \Psi^{-1}(1 \otimes \mathbf{m}).$$

This means that  $\Psi^{-1}(1 \otimes \mathbf{m}) \in \text{Mat}_{r \times 1}(V(M))$ . Thus we have an inequality  $\dim_F V(M) \geq \dim_E M$  and the condition (4) is true.  $\square$

**Corollary 3.11.** *Let  $M$  be a finite-dimensional  $\varphi$ -module over  $E$ . If  $M$  is  $L$ -trivial then  $M$  is étale.*

*Proof.* By Proposition 3.10, there exists a matrix  $\Psi \in \text{GL}_r(L)$  such that  $\sigma \Psi = \Phi \Psi$ . Since  $\sigma$  is injective, we have  $\det \Phi = \sigma(\det \Psi) \det \Psi^{-1} \neq 0$ .  $\square$

Let  $M$  be an  $L$ -trivial  $\varphi$ -module over  $E$ ,  $\mathbf{m} \in \text{Mat}_{r \times 1}(M)$  an  $E$ -basis of  $M$  and  $\Phi \in \text{GL}_r(E)$  a matrix such that  $\varphi \mathbf{m} = \Phi \mathbf{m}$ . By Proposition 3.10, there exists a matrix  $\Psi \in \text{GL}_r(L)$  such that  $\sigma \Psi = \Phi \Psi$ .

**Definition 3.12.** The matrix  $\Psi$  is called a *period matrix* of  $M$  in  $L$  or *fundamental matrix* of  $\Phi$ , and the entries of  $\Psi$  are called *periods* of  $M$  in  $L$ .

Note that  $\Psi' \in \mathrm{GL}_r(L)$  is another fundamental matrix of  $\Phi$  if and only if  $\Psi' = \Psi\delta$  for some  $\delta \in \mathrm{GL}_r(F)$ . Indeed, if  $\sigma(\Psi') = \Phi\Psi'$  then  $\sigma(\Psi^{-1}\Psi') = \sigma(\Psi)^{-1}\sigma(\Psi') = (\Phi\Psi)^{-1}(\Phi\Psi') = \Psi^{-1}\Psi'$ , hence  $\Psi^{-1}\Psi' \in \mathrm{GL}_r(L^\sigma) = \mathrm{GL}_r(F)$ , and the converse is clear.

**Proposition 3.13.** *The period matrix  $\Psi$  of  $M$  is well-defined from  $M$  as an element of  $\mathrm{GL}_r(E) \setminus \mathrm{GL}_r(L) / \mathrm{GL}_r(F)$ .*

*Proof.* Let  $\mathbf{m}' \in \mathrm{Mat}_{r \times 1}(M)$  be another  $E$ -basis of  $M$ ,  $\Phi' \in \mathrm{GL}_r(E)$  and  $\Psi' \in \mathrm{GL}_r(L)$  matrices such that  $\varphi\mathbf{m}' = \Phi'\mathbf{m}'$  and  $\sigma\Psi' = \Phi'\Psi'$ . There exists a matrix  $A \in \mathrm{GL}_r(E)$  which satisfies  $\mathbf{m}' = A\mathbf{m}$ . Then we have  $\varphi\mathbf{m}' = \varphi(A\mathbf{m}) = \sigma(A)\varphi\mathbf{m} = \sigma(A)\Phi\mathbf{m} = \sigma(A)\Phi A^{-1}\mathbf{m}'$ . Thus  $\Phi' = \sigma(A)\Phi A^{-1}$ . We also have  $\sigma(A\Psi) = \sigma(A)\sigma(\Psi) = \sigma(A)\Phi\Psi = \Phi'(A\Psi)$ . Hence we conclude that  $\Psi' \in A\Psi \cdot \mathrm{GL}_r(F)$ .  $\square$

**Proposition 3.14.** *The set of entries of  $\Psi^{-1}(1 \otimes \mathbf{m})$  forms an  $F$ -basis of  $V(M)$ .*

*Proof.* By the proof of Proposition 3.10, we have that  $\Psi^{-1}(1 \otimes \mathbf{m}) \in \mathrm{Mat}_{r \times 1}(V(M))$ . Since  $\dim_F V(M) = \dim_E M = r$ , this is an  $F$ -basis of  $V(M)$ .  $\square$

**Proposition 3.15.** *The  $\varphi$ -module  $(E, \sigma)$  is  $L$ -trivial.*

*Proof.* We have equalities  $V(E) = (L \otimes_E E)^\varphi = L^\sigma = F$ . Therefore  $\dim_F V(E) = 1 = \dim_E E$ .  $\square$

**Proposition 3.16.** *Let  $M$  and  $N$  be  $L$ -trivial  $\varphi$ -modules. Then  $M \oplus N$ ,  $M \otimes N$  and  $\mathrm{Hom}(M, N)$  are also  $L$ -trivial.*

*Proof.* Let  $\mathbf{m} \in \mathrm{Mat}_{r \times 1}(E)$  be an  $E$ -basis of  $M$ ,  $\Phi_M \in \mathrm{GL}_r(E)$  the matrix such that  $\varphi\mathbf{m} = \Phi_M\mathbf{m}$  and  $\Psi_M \in \mathrm{GL}_r(L)$  a matrix which satisfies  $\sigma\Psi_M = \Phi_M\Psi_M$ . We also set  $\mathbf{n} \in \mathrm{Mat}_{s \times 1}(E)$  an  $E$ -basis of  $N$  and  $\Phi_N \in \mathrm{GL}_s(E)$ ,  $\Psi_N \in \mathrm{GL}_s(L)$  matrices which satisfy  $\varphi\mathbf{n} = \Phi_N\mathbf{n}$  and  $\sigma\Psi_N = \Phi_N\Psi_N$ . We set

$$\mathbf{m} \oplus \mathbf{n} := \begin{bmatrix} \mathbf{m} \\ \mathbf{n} \end{bmatrix}, \Phi_M \oplus \Phi_N := \begin{bmatrix} \Phi_M & 0 \\ 0 & \Phi_N \end{bmatrix} \text{ and } \Psi_M \oplus \Psi_N := \begin{bmatrix} \Psi_M & 0 \\ 0 & \Psi_N \end{bmatrix}.$$

Then it is clear that  $\mathbf{m} \oplus \mathbf{n}$  is an  $E$ -basis of  $M \oplus N$ ,  $\varphi(\mathbf{m} \oplus \mathbf{n}) = (\Phi_M \oplus \Phi_N)(\mathbf{m} \oplus \mathbf{n})$  and  $\sigma(\Psi_M \oplus \Psi_N) = (\Phi_M \oplus \Phi_N)(\Psi_M \oplus \Psi_N)$ . Therefore  $M \oplus N$  is  $L$ -trivial.

Set  $\mathbf{m} \otimes \mathbf{n}$  to be an  $E$ -basis of  $M \otimes N$  naturally obtained from  $\mathbf{m}$  and  $\mathbf{n}$ . Let  $\Phi_M \otimes \Phi_N$  be the Kronecker product of  $\Phi_M$  and  $\Phi_N$ , and  $\Psi_M \otimes \Psi_N$  be the Kronecker product of  $\Psi_M$  and  $\Psi_N$ . Then it is clear that  $\varphi(\mathbf{m} \otimes \mathbf{n}) = (\Phi_M \otimes \Phi_N)(\mathbf{m} \otimes \mathbf{n})$  and  $\sigma(\Psi_M \otimes \Psi_N) = (\Phi_M \otimes \Phi_N)(\Psi_M \otimes \Psi_N)$ . Therefore  $M \otimes N$  is  $L$ -trivial.

Let  $\mathbf{m}^\vee$  be the dual basis of  $\mathbf{m}$  for  $M^\vee$ . Then we have equalities  $\varphi\mathbf{m}^\vee = (\Phi_M^{-1})^{\mathrm{tr}}\mathbf{m}^\vee$  and  $\sigma(\Psi_M^{-1})^{\mathrm{tr}} = (\Phi_M^{-1})^{\mathrm{tr}}(\Psi_M^{-1})^{\mathrm{tr}}$ , where  $A^{\mathrm{tr}}$  is the transpose of a matrix  $A$ . Therefore  $M^\vee$  is  $L$ -trivial. Since  $\Phi M_E^{\mathrm{ét}}$  is a rigid tensor category, we have an isomorphism  $M^\vee \otimes N \cong \mathrm{Hom}(M, N)$ . Therefore  $\mathrm{Hom}(M, N)$  is  $L$ -trivial.  $\square$

**Proposition 3.17.** *Let  $0 \rightarrow M' \rightarrow M \rightarrow M'' \rightarrow 0$  be an exact sequence in  $\Phi M_E$ . If  $M$  is  $L$ -trivial, then  $M'$  and  $M''$  are also  $L$ -trivial.*



*Proof.* Let  $\kappa : L \otimes_F V(M) \rightarrow L \otimes_F V(M'')$  be the natural map and  $\iota'' : \text{im}(\kappa) \rightarrow L \otimes_E M''$  be the restriction of the map  $\iota_{M''}$ . Then we have the commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & L \otimes_F V(M') & \longrightarrow & L \otimes_F V(M) & \longrightarrow & \text{im}(\kappa) \longrightarrow 0 \\ & & \downarrow \iota_{M'} & & \downarrow \iota_M & & \downarrow \iota'' \\ 0 & \longrightarrow & L \otimes_E M' & \longrightarrow & L \otimes_E M & \longrightarrow & L \otimes_E M'' \longrightarrow 0, \end{array}$$

where the rows are exact,  $\iota_M$  is an isomorphism and  $\iota_{M'}, \iota''$  are injective. Then we have that  $\iota_{M'}$  and  $\iota''$  are isomorphism by a diagram chasing. Hence  $M'$  and  $M''$  are  $L$ -trivial.  $\square$

**Proposition 3.18.** *The category  $\Phi M_E^L$  is a rigid abelian  $F$ -linear tensor category.*

*Proof.* By Proposition 3.5 and Corollary 3.11, it is enough to show that the category  $\Phi M_E^L$  is closed under direct sum, subquotient, tensor product and internal hom, and has an identity object for the tensor product. By Propositions 3.15, 3.16 and 3.17, these are true.  $\square$

By Corollary 3.8, we can consider  $V$  as a functor from  $\Phi M_E^L$  to the category of finite-dimensional  $F$ -vector spaces  $\mathbf{Vec}(F)$ .

**Proposition 3.19.** *The functor  $V : \Phi M_E^L \rightarrow \mathbf{Vec}(F)$  is  $F$ -linear and exact.*

*Proof.* It is clear that  $V$  is  $F$ -linear. Let  $0 \rightarrow M' \rightarrow M \rightarrow M'' \rightarrow 0$  be an exact sequence in  $\Phi M_E^L$ . It is clear that  $0 \rightarrow V(M') \rightarrow V(M) \rightarrow V(M'') \rightarrow 0$  is exact. This means that  $\dim_F V(M) \leq \dim_F V(M') + \dim_F V(M'')$ . On the other hand, we have  $\dim_F V(M) = \dim_E M = \dim_E M' + \dim_E M'' \geq \dim_F V(M') + \dim_F V(M'')$ .  $\square$

**Proposition 3.20.** *The functor  $V : \Phi M_E^L \rightarrow \mathbf{Vec}(F)$  is faithful.*

*Proof.* Let  $M$  and  $N$  be  $L$ -trivial  $\varphi$ -modules and  $\phi : M \rightarrow N$  a morphism in  $\Phi M_E^L$ . Then we have an exact sequence

$$0 \longrightarrow V(\ker \phi) \longrightarrow V(M) \xrightarrow{V(\phi)} V(N).$$

Therefore if  $V(\phi) = 0$  then  $V(\ker \phi) = V(M)$ . Since  $M$  is  $L$ -trivial, we have an inequality  $\dim_E \ker \phi \geq \dim_F V(\ker \phi) = \dim_F V(M) = \dim_E M$ . This means that  $\ker \phi = M$  and  $\phi = 0$ .  $\square$

**Proposition 3.21.** *Let  $M$  and  $N$  be  $L$ -trivial  $\varphi$ -modules. Then there exists a natural isomorphism  $V(M) \otimes_F V(N) \rightarrow V(M \otimes N)$ . The functor  $V : \Phi M_E^L \rightarrow \mathbf{Vec}(F)$  is a tensor functor with respect to this isomorphism.*

*Proof.* It is clear that there exists a natural isomorphism  $(L \otimes_E M) \otimes_L (L \otimes_E N) \cong L \otimes_E (M \otimes N)$  which preserves  $\varphi$ -actions. By Lemma 3.7, the natural map  $V(M) \otimes_F V(N) \rightarrow (L \otimes_E M) \otimes_L (L \otimes_E N)$  is injective. Therefore we have a natural injection

$$V(M) \otimes_F V(N) \hookrightarrow ((L \otimes_E M) \otimes_L (L \otimes_E N))^\varphi \cong (L \otimes_E (M \otimes N))^\varphi = V(M \otimes N).$$

Since  $\dim_F(V(M) \otimes_F V(N)) = \dim_F V(M \otimes N)$ , this map is a bijection. It is clear that this isomorphism is compatible with the associativity and the commutativity of tensor functors. It is also clear that  $V(E) = F$ . Thus the functor  $V$  is a tensor functor ([6], Definition 1.8).  $\square$

Recall that a *neutral Tannakian category* over a field  $k$  is a rigid abelian  $k$ -linear tensor category  $\mathcal{C}$  for which  $k \xrightarrow{\sim} \text{End}(\mathbf{1})$  and there exists an exact faithful  $k$ -linear tensor functor  $\omega : \mathcal{C} \rightarrow \mathbf{Vec}(k)$ , where  $\mathbf{1}$  is the unit object of  $\mathcal{C}$  ([6], Definition 2.19). Any such functor  $\omega$  is said to be a *fiber functor* for  $\mathcal{C}$ .

**Theorem 3.22.** *The category  $\Phi M_E^L$  is a neutral Tannakian category over  $F$ . The functor  $V : \Phi M_E^L \rightarrow \mathbf{Vec}(F)$  is a fiber functor for  $\Phi M_E^L$ .*

*Proof.* By Proposition 3.2, we have  $\text{End}_\varphi(E) \cong E^\sigma = F$ . By Proposition 3.18, the category  $\Phi M_E^L$  is a rigid abelian  $F$ -linear tensor category. By Propositions 3.19, 3.20 and 3.21, the functor  $V : \Phi M_E^L \rightarrow \mathbf{Vec}(F)$  is an exact faithful  $F$ -linear tensor functor.  $\square$

Let  $M$  be an  $L$ -trivial  $\varphi$ -module over  $E$ . We set  $\mathcal{T}_M$  to be the Tannakian subcategory of  $\Phi M_E^L$  generated by  $M$ , and  $V_M : \mathcal{T}_M \rightarrow \mathbf{Vec}(F)$  the restriction of the functor  $V$ . We denote by  $\Gamma_M$  the Tannakian Galois group of  $(\mathcal{T}_M, V_M)$ . For any  $F$ -algebra  $R$ , we define the functor  $V_M^R : \mathcal{T}_M \rightarrow \mathbf{Mod}(R)$  by  $N \mapsto R \otimes_F V(N)$ , where  $\mathbf{Mod}(R)$  is the category of finitely generated  $R$ -modules. Then by the definition of  $\Gamma_M$ , we have

$$\Gamma_M(R) = \text{Aut}^\otimes(V_M^R)$$

where  $\text{Aut}^\otimes(V_M^R)$  is the group of invertible natural transformations from  $V_M^R$  to itself which is compatible with the tensor products. Therefore we have a canonical injection  $\Gamma_M(R) \hookrightarrow \text{GL}(R \otimes_F V(M))$ .

### 3.3 $v$ -adic case

Let  $t$  be a variable and  $v \in \mathbb{F}_q[t]$  a fixed monic irreducible polynomial of degree  $d$ . For any field  $k$  containing  $\mathbb{F}_q$ , we set  $k[t]_v := \varprojlim (k[t]/v^n)$  and  $k(t)_v := \mathbb{F}_q(t) \otimes_{\mathbb{F}_q[t]} k[t]_v$ .

Let  $\sigma$  be the ring endomorphism of  $k[t]$

$$\sum a_i t^i \mapsto \sum a_i^q t^i.$$

Then  $\sigma$  naturally extends to an endomorphism of  $k(t)_v$ , also denoted by  $\sigma$ . Let  $k'$  be a splitting field of  $v$  over  $k$  in  $\bar{k}$ , and we factorize  $v = \prod_{l \in \mathbb{Z}/d} (t - \lambda_l)$  in  $k'[t]$  with  $\lambda_l^q = \lambda_{l+1}$  for all  $l \in \mathbb{Z}/d$ . Then we have  $k'(t)_v = \prod_{l \in \mathbb{Z}/d} k'((t - \lambda_l))$ , and for any  $a = (\sum_i a_{l,i} (t - \lambda_l)^i)_l \in k'(t)_v$ ,

$$\sigma(a) = (\sum a_{l-1,i}^q (t - \lambda_l)^i)_l.$$

**Lemma 3.23.** *For any field  $k$  containing  $\mathbb{F}_q$ , we have  $(k(t)_v)^\sigma = \mathbb{F}_q(t)_v$ .*

*Proof.* Clearly,  $\mathbb{F}_q(t)_v = (\mathbb{F}_q(t)_v)^\sigma \subset (k(t)_v)^\sigma \subset (k'(t)_v)^\sigma$ . By the explicit description of the  $\sigma$ -action as above, we have  $(k'(t)_v)^\sigma = \{(\sum a_{l,i} (t - \lambda_l)^i)_l \in \mathbb{F}_{q^d}(t)_v \mid a_{l,i}^q = a_{l+1,i} \text{ for all } l \text{ and } i\}$ . This set is isomorphic to  $\mathbb{F}_{q^d}((t - \lambda_l))$  via the  $l$ -th projection for any  $l$ . On the other hand, we have  $\mathbb{F}_q(t)_v \cong \mathbb{F}_{q^d}((t - \lambda_l))$ . Thus the above inclusions are all equalities.  $\square$

Fix a field  $K$  containing  $\mathbb{F}_q$  and assume that  $K \cap \overline{\mathbb{F}_q} = \mathbb{F}_q$ . Note that if  $\mathbb{F}_q$  is not algebraically closed in  $K$ , then  $K(t)_v$  may not be a field and the situation becomes more complicated. Thus in this paper, we always assume that  $K \cap \overline{\mathbb{F}_q} = \mathbb{F}_q$ .

**Lemma 3.24.** *The triple  $(\mathbb{F}_q(t)_v, K(t)_v, K^{\text{sep}}(t)_v)$  is  $\sigma$ -admissible.*

*Proof.* Since  $v$  is irreducible in  $K[t]$ ,  $K(t)_v$  is a field. By Lemma 3.23, we have  $\mathbb{F}_q(t)_v = (K(t)_v)^\sigma = (K^{\text{sep}}(t)_v)^\sigma$ . We need to check the separability. Fix an  $l$ . We need to show that  $K^{\text{sep}}((t - \lambda_l))/K(t)_v$  is a separable field extension. It is clear that  $K(t)_v = K'((t - \lambda_l))$  where  $K' = K(\lambda_l)$ . On the other hand,  $K^{\text{sep}}((t - \lambda_l))/K'((t - \lambda_l))$  is separable since  $K^{\text{sep}}/K'$  is separable ([9], Exercise 26.2).  $\square$

Let  $G_K := \text{Gal}(K^{\text{sep}}/K)$  be the absolute Galois group of  $K$ . Then  $G_K$  acts on  $K^{\text{sep}}[t]$  in an obvious way. This action naturally extends to an action on  $K^{\text{sep}}(t)_v$ . For each  $\tau \in G_K$  and  $a = (\sum_i a_{l,i}(t - \lambda_l)^i)_l \in \prod_l K^{\text{sep}}((t - \lambda_l))$ , we have

$$\tau a = (\sum_i \tau a_{l+n,i}(t - \lambda_l)^i)_l,$$

where  $n \in \mathbb{Z}/d$  is an element such that  $\tau|_{\mathbb{F}_{q^d}} = \sigma|_{\mathbb{F}_{q^d}}^{-n}$ . It is clear that this action is compatible with  $\sigma$ .

From now on, we consider  $\varphi$ -modules over the  $\sigma$ -admissible triple  $(\mathbb{F}_q(t)_v, K(t)_v, K^{\text{sep}}(t)_v)$ . Let  $M$  be an étale  $\varphi$ -module over  $K(t)_v$ . The Galois group  $G_K$  acts on  $K^{\text{sep}}(t)_v \otimes M$  continuously by  $\tau \otimes \text{id}$  for each  $\tau \in G_K$ . Since this action is compatible with  $\sigma$ , the  $\mathbb{F}_q(t)_v$ -subspace  $V(M)$  is  $G_K$ -stable. We denote by  $V_K(M)$  this Galois representation. Conversely for any object  $V$  of  $\mathbf{Rep}(G_K, \mathbb{F}_q(t)_v)$ , we set

$$D(V) := (K^{\text{sep}}(t)_v \otimes_{\mathbb{F}_q(t)_v} V)^{G_K},$$

where  $G_K$  acts on  $K^{\text{sep}}(t)_v \otimes_{\mathbb{F}_q(t)_v} V$  by  $\tau \otimes \tau$  for  $\tau \in G_K$ . Then we can define a  $\varphi$ -action on  $D(V)$  by  $\sigma \otimes \text{id}$ .

Let  $M_0$  be an étale  $\varphi$ -module over  $K[t]_v$ . Then we can define an  $\mathbb{F}_q[t]_v$ -representation of  $G_K$

$$V_0(M_0) := (K^{\text{sep}}[t]_v \otimes_{K[t]_v} M_0)^\varphi,$$

where  $\varphi$  acts on  $K^{\text{sep}}[t]_v \otimes_{K[t]_v} M_0$  by  $\sigma \otimes \varphi$  and  $G_K$  acts on  $V_0(M_0)$  by  $\tau \otimes \text{id}$  for  $\tau \in G_K$ . Conversely for any object  $T$  of  $\mathbf{Rep}(G_K, \mathbb{F}_q[t]_v)$ , we set

$$D_0(T) := (K^{\text{sep}}[t]_v \otimes_{\mathbb{F}_q[t]_v} T)^{G_K},$$

where  $G_K$  acts on  $K^{\text{sep}}[t]_v \otimes_{\mathbb{F}_q[t]_v} T$  by  $\tau \otimes \tau$  for  $\tau \in G_K$ . Then we can define a  $\varphi$ -action on  $D_0(T)$  by  $\sigma \otimes \text{id}$ .

**Theorem 3.25** ([8], Appendix). (1) *For any étale  $\varphi$ -module  $M_0$  over  $K[t]_v$ , the natural map*

$$K^{\text{sep}}[t]_v \otimes_{\mathbb{F}_q[t]_v} V_0(M_0) \rightarrow K^{\text{sep}}[t]_v \otimes_{K[t]_v} M_0$$

*is bijective.*

(2) *For any  $\mathbb{F}_q[t]_v[G_K]$ -module  $T$  of finite type over  $\mathbb{F}_q[t]_v$ , the natural map*

$$K^{\text{sep}}[t]_v \otimes_{K[t]_v} D_0(T) \rightarrow K^{\text{sep}}[t]_v \otimes_{\mathbb{F}_q[t]_v} T$$

*is bijective and the  $\varphi$ -module  $D_0(T)$  is étale.*

(3) *The functor  $V_0 : \Phi M_{K[t]_v}^{\text{ét}} \rightarrow \mathbf{Rep}(G_K, \mathbb{F}_q[t]_v)$  is a tensor equivalence, with a quasi-inverse  $D_0 : \mathbf{Rep}(G_K, \mathbb{F}_q[t]_v) \rightarrow \Phi M_{\mathbb{F}_q[t]_v}^{\text{ét}}$ .*

For any  $\varphi$ -module  $M_0$  over  $K[t]_v$ , we can define a  $\varphi$ -action on  $K(t)_v \otimes_{K[t]_v} M_0$  by  $\sigma \otimes \varphi$ .

**Theorem 3.26.** (1) A  $\varphi$ -module  $M$  over  $K(t)_v$  is  $K^{\text{sep}}(t)_v$ -trivial if and only if there exists a subspace  $M_0$  of  $M$  which is an étale  $\varphi$ -module over  $K[t]_v$  such that  $M = K(t)_v \otimes_{K[t]_v} M_0$ .

(2) For any object  $V$  in  $\mathbf{Rep}(G_K, \mathbb{F}_q(t)_v)$ , the  $\varphi$ -module  $D(V)$  is  $K^{\text{sep}}(t)_v$ -trivial.

(3) The functor  $V : \Phi M_{K(t)_v}^{K^{\text{sep}}(t)_v} \rightarrow \mathbf{Rep}(G_K, \mathbb{F}_q(t)_v)$  is a tensor equivalence, with a quasi-inverse  $D : \mathbf{Rep}(G_K, \mathbb{F}_q(t)_v) \rightarrow \Phi M_{K(t)_v}^{K^{\text{sep}}(t)_v}$ .

*Proof.* Let  $M$  be a  $\varphi$ -module over  $K(t)_v$  such that there exists a subspace  $M_0$  which is an étale  $\varphi$ -module over  $K[t]_v$  and  $M = K(t)_v \otimes_{K[t]_v} M_0$ . Then by Theorem 3.25 (1), we have an isomorphism  $K^{\text{sep}}[t]_v \otimes_{\mathbb{F}_q[t]_v} V_0(M_0) \cong K^{\text{sep}}[t]_v \otimes_{K[t]_v} M_0$ . By tensoring  $K^{\text{sep}}(t)_v$  to the both sides of this isomorphism, we conclude that  $M$  is  $K^{\text{sep}}(t)_v$ -trivial.

Let  $V$  be an object in  $\mathbf{Rep}(G_K, \mathbb{F}_q(t)_v)$ . Then there exists a  $G_K$ -stable  $\mathbb{F}_q[t]_v$ -lattice  $T$  for  $V$ . It is clear that  $D_0(T)$  is free over  $K[t]_v$  and  $D(V) = K(t)_v \otimes_{K[t]_v} D_0(T)$ . Thus  $D(V)$  is  $K^{\text{sep}}(t)_v$ -trivial from the above argument and this proves (2). By Theorem 3.25 (2), we have an isomorphism  $K^{\text{sep}}(t)_v \otimes_{K(t)_v} D(V) \cong K^{\text{sep}}(t)_v \otimes_{\mathbb{F}_q(t)_v} V$ . By taking the  $\varphi$ -fixed parts of the both sides of this isomorphism, we have an isomorphism  $V_K(D(V)) \cong V$ .

Let  $M$  be a  $K^{\text{sep}}(t)_v$ -trivial  $\varphi$ -module over  $K(t)_v$ . Then we have an isomorphism  $K^{\text{sep}}(t)_v \otimes_{\mathbb{F}_q(t)_v} V_K(M) \cong K^{\text{sep}}(t)_v \otimes_{K(t)_v} M$ . By taking the  $G_K$ -fixed parts of the both sides of this isomorphism, we have an isomorphism  $D(V_K(M)) \cong M$ , and this proves (3). Therefore  $M$  comes from étale  $\varphi$ -module over  $K[t]_v$  and this proves (1).  $\square$

## 4 Frobenius equations

Throughout this section, we fix a  $\sigma$ -admissible triple  $(F, E, L)$ .

**Example 4.1.** The case  $(F, E, L) = (\mathbb{F}_q(t)_v, K(t)_v, K^{\text{sep}}(t)_v)$  is our main example of a  $\sigma$ -admissible triple, where the notation and the  $\sigma$ -action are as in Subsection 3.3.

**Example 4.2.** Let  $(F, E, L) = (\mathbb{F}_q(t)_v, K^{\text{rad}}(t)_v, \bar{K}(t)_v)$  where  $K^{\text{rad}} := \cup_n K^{1/q^n}$ , the maximal radical extension of  $K$  in  $\bar{K}$ . The automorphism of  $K^{\text{rad}}(t)$

$$\sum_i a_i t^i \mapsto \sum_i a_i^{1/q} t^i$$

is naturally extends to an automorphism of  $\bar{K}(t)_v$ . We define  $\sigma$  to be this action. Then  $(\mathbb{F}_q(t)_v, K^{\text{rad}}(t)_v, \bar{K}(t)_v)$  is a  $\sigma$ -admissible triple. Note that, in this case we need to put  $L_l = \bar{K}((t - \lambda_{-l}))$ . Note also that we do not use this type in this paper. However, the  $\sigma$ -action of this type is used in [10] and [5].

### 4.1 The group $\Gamma$

Let  $r$  be a positive integer. Fix matrices  $\Phi = (\Phi_{ij}) \in \text{GL}_r(E)$  and  $\Psi = (\Psi_{ij})_{i,j} \in \text{GL}_r(L)$  such that  $\Psi$  is a fundamental matrix for  $\Phi$ . Thus we have an equation

$$\sigma(\Psi) = \Phi\Psi.$$

This means that the matrices  $\Phi$  and  $\Psi$  come from an  $L$ -trivial  $\varphi$ -module over  $E$ . Since  $L = \prod_l L_l$ , we can write  $\Psi_{ij} = (\Psi_{ijl})_l$  for each  $i$  and  $j$ . We set  $\Psi_l := (\Psi_{ijl})_{i,j} \in \text{GL}_r(L_l)$ . Then we have  $\sigma(\Psi_l) = \Phi\Psi_{l+1}$  for all  $l$ .

Let  $X := (X_{ij})$  be an  $r \times r$  matrix of independent variables  $X_{ij}$ , and set  $\Delta := \det(X)$ . We set  $E[X, \Delta^{-1}] := E[X_{11}, X_{12}, \dots, X_{rr}, \Delta^{-1}]$ . Similarly  $E[\Psi, \Delta(\Psi)^{-1}]$  and  $E[\Psi_l, \Delta(\Psi_l)^{-1}]$  are defined. We define  $E$ -algebra homomorphisms  $\nu : E[X, \Delta^{-1}] \rightarrow L$ ;  $X_{ij} \mapsto \Psi_{ij}$  and  $\nu_l : E[X, \Delta^{-1}] \rightarrow L_l$ ;  $X_{ij} \mapsto \Psi_{ijl}$ . Set  $\mathfrak{p} := \ker \nu$ ,  $\Sigma := E[\Psi, \Delta(\Psi)^{-1}] \cong E[X, \Delta^{-1}]/\mathfrak{p}$ ,  $Z := \operatorname{Spec} \Sigma$ ,  $\mathfrak{p}_l := \ker \nu_l$ ,  $\Sigma_l := E[\Psi_l, \Delta(\Psi_l)^{-1}] \cong E[X, \Delta^{-1}]/\mathfrak{p}_l$  and  $Z_l := \operatorname{Spec} \Sigma_l$ . Then  $Z_l$  are closed subschemes of  $Z$  and  $Z = \cup_l Z_l$ . Let  $\Lambda := \operatorname{Frac}(\Sigma)$  and  $\Lambda_l := \operatorname{Frac}(\Sigma_l)$ , the total rings of fractions.

Set  $\Psi_1 := (\Psi_{ij} \otimes 1)_{i,j}$ ,  $\Psi_2 := (1 \otimes \Psi_{ij})_{i,j}$  and  $\tilde{\Psi} = (\tilde{\Psi}_{ij})_{i,j} := \Psi_1^{-1} \Psi_2$  in  $\operatorname{GL}_r(L \otimes_E L)$ . Since  $L \otimes_E L = \prod_{l,m} L_l \otimes_E L_m$ , we can write  $\tilde{\Psi}_{ij} = (\tilde{\Psi}_{ijlm})_{l,m}$  with  $\tilde{\Psi}_{ijlm} \in L_l \otimes_E L_m$  for each  $i$  and  $j$ . We define  $F$ -algebra homomorphisms  $\mu : F[X, \Delta^{-1}] \rightarrow L \otimes_E L$ ;  $X_{ij} \mapsto \tilde{\Psi}_{ij}$  and  $\mu_{lm} : F[X, \Delta^{-1}] \rightarrow L_l \otimes_E L_m$ ;  $X_{ij} \mapsto \tilde{\Psi}_{ijlm}$ . Set  $\mathfrak{q} := \ker \mu$ ,  $\Gamma := \operatorname{Spec} F[X, \Delta^{-1}]/\mathfrak{q}$ ,  $\mathfrak{q}_{lm} := \ker \mu_{lm}$  and  $\Gamma_{lm} := \operatorname{Spec} F[X, \Delta^{-1}]/\mathfrak{q}_{lm}$ . Then  $\Gamma_{lm}$  are closed subschemes of  $\Gamma$  and  $\Gamma = \cup_{l,m} \Gamma_{lm}$ . By the next lemma, we can set  $\mathfrak{q}_m := \mathfrak{q}_{0,m} = \mathfrak{q}_{1,m+1} = \dots$  and  $\Gamma_m := \Gamma_{0,m} = \Gamma_{1,m+1} = \dots$ .

**Lemma 4.3.** *For any  $l, m \in \mathbb{Z}/d$ , we have  $\mathfrak{q}_{lm} = \mathfrak{q}_{l+1,m+1} = \mathfrak{q}_{l+2,m+2} = \dots$ .*

*Proof.* Let  $\tilde{L}$  be the inductive limit of the inductive system  $L \rightarrow L \rightarrow L \rightarrow \dots$ , where the transition maps are  $\sigma$ . Then  $L$  is a subring of  $\tilde{L}$  and  $\sigma$  is naturally extends to an automorphism of  $\tilde{L}$ . We can define a  $\sigma$ -action on  $\tilde{L} \otimes_E \tilde{L}$  by  $\sigma \otimes \sigma$ . This is an isomorphism and  $L \otimes_E L$  is stable under this action. Thus we obtain an injective endomorphism  $\sigma$  of  $L \otimes_E L$ . It is clear that  $\sigma(L_l \otimes_E L_m) \subset L_{l+1} \otimes_E L_{m+1}$ .

Write  $\Psi_1 = (\Psi_{1,lm})_{l,m}$  and  $\Psi_2 = (\Psi_{2,lm})_{l,m}$  with  $\Psi_{i,lm} \in \operatorname{GL}_r(L_l \otimes_E L_m)$ , and set  $\tilde{\Psi}^{(lm)} := (\tilde{\Psi}_{ijlm})_{i,j} \in \operatorname{GL}_r(L_l \otimes_E L_m)$  for each  $l$  and  $m$ . Then we obtain the equality  $\sigma(\tilde{\Psi}^{(lm)}) = \sigma(\Psi_{1,lm})^{-1} \sigma(\Psi_{2,lm}) = (\Phi \Psi_{1,l+1,m+1})^{-1} (\Phi \Psi_{2,l+1,m+1}) = \tilde{\Psi}^{(l+1,m+1)}$ . For any  $h(X) \in F[X, \Delta^{-1}]$ , we have  $h(\tilde{\Psi}^{(lm)}) = 0$  if and only if  $h(\tilde{\Psi}^{(l+1,m+1)}) = 0$  since  $\sigma(h(\tilde{\Psi}^{(lm)})) = h(\tilde{\Psi}^{(l+1,m+1)})$  and  $\sigma$  is injective on  $L \otimes_E L$ . This proves the lemma.  $\square$

For any  $h(X) \in L[X, \Delta^{-1}]$ , we denote by  $h^\sigma(X)$  the polynomial obtained by applying  $\sigma$  to the coefficients of  $h(X)$ . We define two endomorphisms

$$\sigma_0 : L[X, \Delta^{-1}] \rightarrow L[X, \Delta^{-1}]; \quad h(X) \mapsto h^\sigma(X),$$

$$\sigma_1 : L[X, \Delta^{-1}] \rightarrow L[X, \Delta^{-1}]; \quad h(X) \mapsto h^\sigma(\Phi X).$$

Then  $\sigma_0(L_l[X, \Delta^{-1}]) \subset L_{l+1}[X, \Delta^{-1}]$  and  $\sigma_1(L_l[X, \Delta^{-1}]) \subset L_{l+1}[X, \Delta^{-1}]$ .

**Lemma 4.4.** *We have  $\sigma_1 \mathfrak{p} \subset \mathfrak{p}$ ,  $\sigma_1 \mathfrak{p}_l \subset \mathfrak{p}_{l+1}$ ,  $\sigma_0 \mathfrak{q} = \mathfrak{q}$ ,  $\sigma_0 \mathfrak{q}_m = \mathfrak{q}_m$ ,  $\sigma \nu = \nu \sigma_1|_{E[X, \Delta^{-1}]}$  and  $\sigma \nu_l = \nu_{l+1} \sigma_1|_{E[X, \Delta^{-1}]}$  for each  $l$  and  $m$ .*

*Proof.* For any  $h(X) \in E[X, \Delta^{-1}]$ , we have  $\nu_{l+1}(\sigma_1(h(X))) = (\sigma_1 h)(\Psi_{l+1}) = h^\sigma(\Phi \Psi_{l+1}) = h^\sigma(\sigma \Psi_l) = \sigma(h(\Psi_l)) = \sigma(\nu_l(h(X)))$ . If  $h \in \mathfrak{p}_l$ , then  $(\sigma_1 h)(\Psi_{l+1}) = \sigma(h(\Psi_l)) = 0$ , and hence  $\sigma_1 h \in \mathfrak{p}_{l+1}$ . Since  $\mathfrak{q}_l \subset F[X, \Delta^{-1}]$  and  $\sigma_0|_{F[X, \Delta^{-1}]} = \operatorname{id}$ , we have  $\sigma_0 \mathfrak{q}_l = \mathfrak{q}_l$ . The other assertions are proved similarly.  $\square$

For any ring homomorphism  $R \rightarrow S$  and any ideal  $\mathfrak{a} \subset R[X, \Delta^{-1}]$ , we set  $\mathfrak{a}_S := \mathfrak{a} \cdot S[X, \Delta^{-1}]$ , the extension ideal of  $\mathfrak{a}$ .

**Lemma 4.5.** *There exists a bijection between the set of ideals of  $F[X, \Delta^{-1}]$  and the set of ideals of  $L[X, \Delta^{-1}]$  which are  $\sigma_0$ -stable, via the extension and the restriction of ideals.*

*Proof.* For any ideal  $\mathfrak{a} \subset F[X, \Delta^{-1}]$ , it is clear that  $\sigma_0 \mathfrak{a}_L \subset \mathfrak{a}_L$ . Because of the faithfully flatness of the inclusion  $F[X, \Delta^{-1}] \hookrightarrow L[X, \Delta^{-1}]$ , we have  $\mathfrak{a} = \mathfrak{a}_L \cap F[X, \Delta^{-1}]$ .

Conversely, we take any ideal  $\mathfrak{b} \subset L[X, \Delta^{-1}]$  with  $\sigma_0 \mathfrak{b} \subset \mathfrak{b}$ , and set  $\mathfrak{a} := \mathfrak{b} \cap F[X, \Delta^{-1}]$ . It is clear that  $\mathfrak{b} \supset \mathfrak{a}_L$ ; thus we need to show that the converse inclusion  $\mathfrak{b} \subset \mathfrak{a}_L$ .

Take an  $F$ -basis  $(g_i)_{i \in I}$  of  $F[X, \Delta^{-1}]$ . Then this is an  $L$ -basis of  $L[X, \Delta^{-1}]$ . For each  $h = \sum_i b_i g_i \in L[X, \Delta^{-1}]$ , we set  $\text{supp}(h) := \{i \in I \mid b_i \neq 0\}$  and  $l(h) := \#\text{supp}(h)$ . We take  $h \in \mathfrak{b}$  and show that  $h \in \mathfrak{a}_L$  by induction on  $l(h)$ . If  $l(h) = 0$ , then  $h = 0 \in \mathfrak{a}_L$ . Now suppose that  $l(h) > 0$ , and assume that if  $\tilde{h} \in \mathfrak{b}$  and  $l(\tilde{h}) < l(h)$  then  $\tilde{h} \in \mathfrak{a}_L$ . Let  $e_l \in L$  be the element such that the  $l$ -th component is one and the other components are all zero. Then it is clear that  $\sigma e_l = e_{l+1}$ . We write  $h = \sum_i b_i g_i$  and take  $i_1$  such that  $b_{i_1} \neq 0$ . Take  $l_0$  such that the  $l_0$ -th component of  $b_{i_1}$  is non-zero. Then there exists an element  $b' \in L$  such that  $b' b_{i_1} = e_{l_0}$ . Since  $\mathfrak{b}$  is an ideal and  $\sigma_0$ -stable, we have

$$\mathfrak{b} \ni \sum_{j=0}^{d-1} \sigma_0^j(b'h) = \sum_{j=0}^{d-1} \sigma_0^j(b' \sum_i b_i g_i) = \sum_i \sum_{j=0}^{d-1} \sigma_0^j(b' b_i) g_i =: \sum_i c_i g_i =: h',$$

$c_{i_1} = \sum_j \sigma_0^j(b' b_{i_1}) = 1$  and  $\text{supp}(h') \subset \text{supp}(h)$ . Therefore  $h - b_{i_1} h' \in \mathfrak{b}$  and  $l(h - b_{i_1} h') < l(h)$ . By induction hypothesis, we have  $h - b_{i_1} h' \in \mathfrak{a}_L$ . Hence it is enough to show that  $h' \in \mathfrak{a}_L$ . If  $c_i \in F$  for all  $i$ , then  $h' \in \mathfrak{b} \cap F[X, \Delta^{-1}] = \mathfrak{a} \subset \mathfrak{a}_L$ . If  $\#L_l \leq 3$  for some (hence for all)  $l$ , we can write the  $\sigma$  action on  $L$  by  $(x_l)_l \mapsto (x_{l-1})_l$ . Hence  $c_i \in F$  for all  $i$ . Thus we assume that  $c_{i_2} \in L \setminus F$  for some  $i_2$  and  $\#L_l \geq 4$  for all  $l$ .

We claim that, we can construct an element  $\bar{h} = \sum_i a_i g_i \in \mathfrak{a}_L$  which has the properties that  $\text{supp}(\bar{h}) \subset \text{supp}(h')$  and  $a_{i_1} = 1$ . We first show that the claim implies  $h' \in \mathfrak{a}_L$ . Since  $h' - \bar{h} \in \mathfrak{b}$  and  $l(h' - \bar{h}) < l(h') \leq l(h)$ , we have  $h' - \bar{h} \in \mathfrak{a}_L$  by induction hypothesis. Thus we have  $h' \in \mathfrak{a}_L$  since  $\bar{h} \in \mathfrak{a}_L$ .

Now we prove the claim. First, we construct an element  $\bar{h} = \sum_i a_i g_i \in \mathfrak{b}$  which has the properties that  $\text{supp}(\bar{h}) \subset \text{supp}(h')$ ,  $a_{i_1} = 1$ ,  $a_{i_2} \in L^\times$  and  $\sigma(a_{i_2}^{-1}) - a_{i_2}^{-1} \in L^\times$ . If  $d = 1$ , then we can take  $\bar{h} = h'$  since  $L$  is a field and  $\sigma(c_{i_2}^{-1}) - c_{i_2}^{-1} \neq 0$ . Thus we suppose that  $d \geq 2$ . Since  $c_{i_2} = (c_{i_2, l})_l \notin F$ , there exists an  $l_0$  such that  $\sigma c_{i_2, l_0-1} \neq c_{i_2, l_0}$ . Thus there is an element  $c' \in L$  such that  $c'(\sigma c_{i_2} - c_{i_2}) = e_{l_0}$ . We set

$$h'' := \sum_i \alpha_i g_i := \sum_i \sum_{j=0}^{d-1} \sigma_0^j(c'(\sigma c_i - c_i)) g_i = \sum_{j=0}^{d-1} \sigma_0^j(c'(\sigma_0 h' - h')) \in \mathfrak{b}.$$

Then we have  $\alpha_{i_1} = 0$ ,  $\alpha_{i_2} = 1$  and  $\text{supp}(h'') \subset \text{supp}(h')$ . For  $f = (f_l)_l \in L$ , consider the element  $\bar{h} := h' - f h'' = g_{i_1} + (c_{i_2, l} - f_l) l g_{i_2} + \dots \in \mathfrak{b}$ . For any  $x = (x_l)_l \in L^\times$ ,  $\sigma x^{-1} - x^{-1} \in L^\times$  if and only if  $\sigma x_l \neq x_{l+1}$  for all  $l$ . Therefore, it is enough to take an element  $f$  such that  $c_{i_2, l} \neq f_l$  and  $\sigma(c_{i_2, l-1} - f_{l-1}) \neq c_{i_2, l} - f_l$  for all  $l$ . Since  $\#L_l \geq 4$ , we can take  $(f_l)_l$  inductively so that  $f_1 \in L_1 \setminus \{c_{i_2, 1}\}$ ,  $f_l \in L_l \setminus \{c_{i_2, l}, c_{i_2, l} - \sigma(c_{i_2, l-1} - f_{l-1})\}$  for  $2 \leq l < d$  and  $f_d \in L_d \setminus (\{c_{i_2, d}, c_{i_2, d} - \sigma(c_{i_2, d-1} - f_{d-1})\} \cup \sigma^{-1}(\sigma(c_{i_2, d}) - c_{i_2, 1} + f_1))$ . Then such  $(f_l)_l$  satisfies the above properties. Next, we show that  $\bar{h} \in \mathfrak{a}_L$ . Since  $\sigma_0 \bar{h} - \bar{h} \in \mathfrak{b}$  and  $l(\sigma_0 \bar{h} - \bar{h}) < l(\bar{h}) \leq l(h)$ , we have  $\sigma_0 \bar{h} - \bar{h} \in \mathfrak{a}_L$  by induction hypothesis. Similarly, we can show that  $\sigma_0(a_{i_2}^{-1} \bar{h}) - a_{i_2}^{-1} \bar{h} \in \mathfrak{a}_L$ . Therefore we have  $(\sigma(a_{i_2}^{-1}) - a_{i_2}^{-1}) \bar{h} = (\sigma_0(a_{i_2}^{-1} \bar{h}) - a_{i_2}^{-1} \bar{h}) - \sigma(a_{i_2}^{-1})(\sigma_0 \bar{h} - \bar{h}) \in \mathfrak{a}_L$ . Since  $(\sigma(a_{i_2}^{-1}) - a_{i_2}^{-1}) \in L^\times$ , we have  $\bar{h} \in \mathfrak{a}_L$ .  $\square$

**Lemma 4.6.** *The map  $\prod_l \mathfrak{b}_l \mapsto (\mathfrak{b}_l)_l$  is a bijection between the set of ideals of  $L[X, \Delta^{-1}]$  which are  $\sigma_0$ -stable, and the set of families  $(\mathfrak{b}_l)_l$  where  $\mathfrak{b}_l$  is an ideal of  $L_l[X, \Delta^{-1}]$  and  $\sigma_0 \mathfrak{b}_l \subset \mathfrak{b}_{l+1}$  for all  $l$ .*

*Proof.* This is clear.  $\square$

**Lemma 4.7.** *For each  $l$ , we give an ideal  $\mathfrak{b}_l \subset L_l[X, \Delta^{-1}]$  such that  $\sigma_0 \mathfrak{b}_l \subset \mathfrak{b}_{l+1}$ . Then the restriction  $\mathfrak{b}_l \cap F[X, \Delta^{-1}]$  is independent of  $l$ . The same is also true if we replace  $L_l$  by  $\Sigma_l$ .*

*Proof.* We only prove the case of  $L_l$ . Let  $\iota$  be the natural injection  $F[X, \Delta^{-1}] \hookrightarrow L[X, \Delta^{-1}]$  and  $\pi_l$  the natural projection  $L[X, \Delta^{-1}] \twoheadrightarrow L_l[X, \Delta^{-1}]$  for each  $l$ . For each  $l$ ,  $\sigma_0$  induces a morphism  $L_l[X, \Delta^{-1}] \rightarrow L_{l+1}[X, \Delta^{-1}]$ ; we also denote this by  $\sigma_0$ . Then we have an equality  $\pi_{l+1}\iota = \sigma_0\pi_l\iota$ . For any  $h \in \mathfrak{b}_l \cap F[X, \Delta^{-1}] = (\pi_l\iota)^{-1}\mathfrak{b}_l$ , we have  $\pi_{l+1}(\iota(h)) = \sigma_0(\pi_l(\iota(h))) \in \sigma_0\mathfrak{b}_l \subset \mathfrak{b}_{l+1}$ . Hence  $h \in (\pi_{l+1}\iota)^{-1}\mathfrak{b}_{l+1} = \mathfrak{b}_{l+1} \cap F[X, \Delta^{-1}]$ . Therefore we obtain  $\mathfrak{b}_l \cap F[X, \Delta^{-1}] \subset \mathfrak{b}_{l+1} \cap F[X, \Delta^{-1}]$ . Since the index set  $\mathbb{Z}/d$  is a finite cyclic group, this inclusion is an equality.  $\square$

For any ring  $R$ , we denote by  $\mathrm{GL}_{r/R}$  the  $R$ -group scheme of  $r \times r$  invertible matrices.

**Proposition 4.8.** (1) *Let  $\phi : Z_L \rightarrow \mathrm{GL}_{r/L}$  be the morphism of affine  $L$ -schemes defined by  $u \mapsto \Psi^{-1}u$  for any  $L$ -algebra  $S$  and any  $S$ -valued point  $u \in Z(S)$ . Then  $\phi$  factors through an isomorphism  $\phi' : Z_L \rightarrow \Gamma_L$  of affine  $L$ -schemes.*

(2) *For any  $l$  and  $m$ , let  $\phi_{lm} : Z_{m,L_l} \rightarrow \mathrm{GL}_{r/L_l}$  be the morphism of affine  $L_l$ -schemes defined by  $u \mapsto \Psi_l^{-1}u$  for any  $L_l$ -algebra  $S$  and any  $S$ -valued point  $u \in Z_m(S)$ . Then  $\phi_{lm}$  factors through an isomorphism  $\phi'_{lm} : Z_{m,L_l} \rightarrow \Gamma_{m-l,L_l}$  of affine  $L_l$ -schemes.*

$$\begin{array}{ccc}
 Z_L & \xrightarrow{\phi; u \mapsto \Psi^{-1}u} & \mathrm{GL}_{r/L} \\
 \searrow \phi' & \nearrow \text{natural} & \\
 & \Gamma_L &
 \end{array}
 \qquad
 \begin{array}{ccc}
 Z_{m,L_l} & \xrightarrow{\phi_{lm}; u \mapsto \Psi_l^{-1}u} & \mathrm{GL}_{r/L_l} \\
 \searrow \phi'_{lm} & \nearrow \text{natural} & \\
 & \Gamma_{m-l,L_l} &
 \end{array}$$

*Proof.* We prove only (2). Then (1) can be proved by the same argument. We define two  $L_l$ -algebra homomorphisms:

$$(4.1) \quad \alpha_l : L_l[X, \Delta^{-1}] \rightarrow L_l[X, \Delta^{-1}]; \quad X \mapsto \Psi_l^{-1}X,$$

$$(4.2) \quad \bar{\alpha}_{lm} : L_l[X, \Delta^{-1}] \xrightarrow{\alpha_l} L_l[X, \Delta^{-1}] \twoheadrightarrow L_l[X, \Delta^{-1}]/\mathfrak{p}_{m,L_l} = L_l \otimes_E E[X, \Delta^{-1}]/\mathfrak{p}_m.$$

Then  $\phi_{lm}$  corresponds to  $\bar{\alpha}_{lm}$  on the level of coordinate rings. Thus it is enough to show that  $\alpha_l^{-1}\mathfrak{p}_{m,L_l} = \mathfrak{q}_{m-l,L_l}$ .

For any  $h(X) \in L_l[X, \Delta^{-1}]$ , we have

$$\sigma_1 \alpha_l h = \sigma_1(h(\Psi_l^{-1}X)) = h^\sigma((\sigma\Psi_l^{-1})\Phi X) = h^\sigma(\Psi_{l+1}^{-1}X) = \alpha_{l+1}h^\sigma(X) = \alpha_{l+1}\sigma_0 h.$$

Therefore we have

$$\alpha_{l+1}^{-1}\sigma_1 = \sigma_0\alpha_l^{-1}.$$

Since  $\sigma_1\mathfrak{p}_m \subset \mathfrak{p}_{m+1}$  by Lemma 4.4, we have an inclusion  $\sigma_0\alpha_l^{-1}\mathfrak{p}_{m,L_l} = \alpha_{l+1}^{-1}\sigma_1\mathfrak{p}_{m,L_l} \subset \alpha_{l+1}^{-1}\mathfrak{p}_{m+1,L_{l+1}}$ . Replacing  $m$  by  $m+l$ , we obtain  $\sigma_0\alpha_l^{-1}\mathfrak{p}_{m+l,L_l} \subset \alpha_{l+1}^{-1}\mathfrak{p}_{m+l+1,L_{l+1}}$ . We consider the family of ideals  $(\alpha_l^{-1}\mathfrak{p}_{m+l,L_l})_l$ . Then for each  $l$  and  $m$ , we have  $(\alpha_l^{-1}\mathfrak{p}_{m+l,L_l} \cap F[X, \Delta^{-1}])_{L_l} = \alpha_l^{-1}\mathfrak{p}_{m+l,L_l}$  by Lemmas 4.5, 4.6 and 4.7. Again, replacing  $m$  by  $m-l$ , we obtain an equality  $(\alpha_l^{-1}\mathfrak{p}_{m,L_l} \cap F[X, \Delta^{-1}])_{L_l} = \alpha_l^{-1}\mathfrak{p}_{m,L_l}$ .

We consider  $L_l \otimes_E L_m$  as an  $L_l$ -algebra via  $f \mapsto f \otimes 1$ , and define an  $L_l$ -algebra homomorphism  $\tilde{\mu} : L_l[X, \Delta^{-1}] \rightarrow L_l \otimes_E L_m$ ;  $X_{ij} \mapsto 1 \otimes \Psi_{ijm}$ . Then  $\mu_{lm} = \tilde{\mu} \circ \alpha_l|_{F[X, \Delta^{-1}]}$  and the map

$$L_l[X, \Delta^{-1}] \twoheadrightarrow L_l[X, \Delta^{-1}]/\mathfrak{p}_{m, L_l} = L_l \otimes_E E[X, \Delta^{-1}]/\mathfrak{p}_m \xrightarrow{\text{id} \otimes \mu_m} L_l \otimes_E L_m$$

coincides with  $\tilde{\mu}$ . Therefore,

$$\mathfrak{q}_{m-l} = \mathfrak{q}_{l, m} = \ker \mu_{l, m} = \alpha_l|_{F[X, \Delta^{-1}]}^{-1}(\ker \tilde{\mu}) = \alpha_l^{-1} \mathfrak{p}_{m, L_l} \cap F[X, \Delta^{-1}].$$

Thus we have  $\mathfrak{q}_{m-l, L_l} = (\alpha_l^{-1} \mathfrak{p}_{m, L_l} \cap F[X, \Delta^{-1}])_{L_l} = \alpha_l^{-1} \mathfrak{p}_{m, L_l}$ .  $\square$

**Lemma 4.9.** (1) *The ideal  $\mathfrak{p} \subsetneq E[X, \Delta^{-1}]$  is maximal among the proper  $\sigma_1$ -invariant ideals.*

(2) *The family of ideals  $(\mathfrak{p}_l)_l$  is maximal among the families of proper ideals  $(\mathfrak{m}_l)_l$  of  $E[X, \Delta^{-1}]$  which satisfies  $\sigma_1 \mathfrak{m}_l \subset \mathfrak{m}_{l+1}$  for all  $l$ .*

*Proof.* We prove only (2). Then (1) can be proved by the same argument. Let  $(\mathfrak{m}_l)_l$  be a family of proper ideals of  $E[X, \Delta^{-1}]$  such that  $\mathfrak{p}_l \subset \mathfrak{m}_l$  and  $\sigma_1 \mathfrak{m}_l \subset \mathfrak{m}_{l+1}$  for all  $l$ . Let  $\alpha_l$  be the homomorphism (4.1). We consider the family of ideals  $(\alpha_l^{-1} \mathfrak{m}_{l, L_l})_l$ . Since  $\sigma_0 \alpha_l^{-1} \mathfrak{m}_{l, L_l} = \alpha_{l+1}^{-1} \sigma_1 \mathfrak{m}_{l, L_l} \subset \alpha_{l+1}^{-1} \mathfrak{m}_{l+1, L_{l+1}}$ , we can apply Lemma 4.6 to  $(\alpha_l^{-1} \mathfrak{m}_{l, L_l})_l$ . Then  $\alpha_l^{-1} \mathfrak{m}_{l, L_l} \cap F[X, \Delta^{-1}]$  is independent of  $l$ , and we take a maximal ideal  $\mathfrak{a} \subset F[X, \Delta^{-1}]$  which contains this ideal. We also have  $(\alpha_l^{-1} \mathfrak{m}_{l, L_l} \cap F[X, \Delta^{-1}])_{L_l} = \alpha_l^{-1} \mathfrak{m}_{l, L_l}$  by Lemmas 4.5, 4.6 and 4.7. Thus we obtain an inclusion  $\alpha_l^{-1} \mathfrak{m}_{l, L_l} \subset \mathfrak{a}_{L_l}$ . We put  $M := F[X, \Delta^{-1}]/\mathfrak{a}$  and define a morphism

$$\pi_l : E[X, \Delta^{-1}] \hookrightarrow L_l[X, \Delta^{-1}] \xrightarrow{\alpha_l^{-1}} L_l[X, \Delta^{-1}] \xrightarrow{\rho_l} L_l[X, \Delta^{-1}]/\mathfrak{a}_{L_l} \xrightarrow{\beta_l} L_l \otimes_F M,$$

where  $\rho_l$  is the natural projection and  $\beta_l : L_l[X, \Delta^{-1}]/\mathfrak{a}_{L_l} \cong L_l \otimes_F F[X, \Delta^{-1}]/\mathfrak{a} = L_l \otimes_F M$ . Then we have  $\mathfrak{m}_l \subset \ker \pi_l$ .

We define a  $\sigma$ -action on  $L \otimes_F M$  by  $\sigma \otimes \text{id}$ . In  $\text{GL}_r(L_{l+1} \otimes_F M)$ , we have

$$\begin{aligned} \sigma(\pi_l(X)) &= \sigma(\beta_l(\rho_l(\alpha_l^{-1}(X)))) = \beta_{l+1}(\rho_{l+1}(\sigma_0(\alpha_l^{-1}(X)))) = \beta_{l+1}(\rho_{l+1}(\alpha_{l+1}^{-1}(\sigma_1(X)))) \\ &= \pi_{l+1}(\sigma_1(X)) = \pi_{l+1}(\Phi X) = \Phi \pi_{l+1}(X). \end{aligned}$$

Set  $\pi(X) := (\pi_l(X))_l \in \text{GL}_r(L \otimes_F M)$ . Then we have  $\sigma(\pi(X)) = \Phi \pi(X)$ . Since  $(L \otimes_F M)^\sigma = M$ , we obtain  $\delta := \pi(X)^{-1} \Psi \in \text{GL}_r((L \otimes_F M)^\sigma) = \text{GL}_r(M)$ . We define a  $\delta$ -action on  $(E \otimes_F M)[X, \Delta^{-1}]$  by  $\delta \cdot h(X) := h(X\delta)$ . We extend  $\pi_l$  to

$$\pi'_l : (E \otimes_F M)[X, \Delta^{-1}] = E[X, \Delta^{-1}] \otimes_F M \xrightarrow{\pi_l \otimes \text{id}} L_l \otimes_F M.$$

Then we have  $\mathfrak{p}_l \otimes_F M \subset \mathfrak{m}_l \otimes_F M \subset \ker \pi'_l = \delta \cdot \ker(\nu_l \otimes \text{id}_M) = \delta \cdot (\mathfrak{p}_l \otimes_F M)$ , where the first equality is proved as follows: For any  $h(X) \in (E \otimes_F M)[X, \Delta^{-1}]$ ,  $(\nu_l \otimes \text{id}_M)(h(X\delta^{-1})) = h(\Psi_l \Psi^{-1} \pi(X)) = h(\pi_l(X)) = \pi'_l(h(X))$ . Thus  $h \in \ker \pi'_l$  is equivalent to  $\delta^{-1} \cdot h \in \ker(\nu_l \otimes \text{id}_M)$ . Since  $(E \otimes_F M)[X, \Delta^{-1}]$  is a noetherian ring,  $\mathfrak{p}_l \otimes_F M \subset \delta \cdot (\mathfrak{p}_l \otimes_F M)$  implies  $\mathfrak{p}_l \otimes_F M = \delta \cdot (\mathfrak{p}_l \otimes_F M)$ . Therefore we have  $\mathfrak{p}_l \otimes_F M = \mathfrak{m}_l \otimes_F M$ . Since  $(E \otimes_F M)[X, \Delta^{-1}]$  is faithfully flat over  $E[X, \Delta^{-1}]$ , we have  $\mathfrak{p}_l = \mathfrak{m}_l$ .  $\square$



**Lemma 4.10.** (1) Let  $\mathfrak{b} \subset \Sigma[X, \Delta^{-1}]$  be an ideal which is  $\sigma_0$ -invariant. Then we have  $\mathfrak{b} = (\mathfrak{b} \cap F[X, \Delta^{-1}])_{\Sigma}$ .

(2) Let  $\mathfrak{b}_l \subset \Sigma_l[X, \Delta^{-1}]$  be ideals which satisfy  $\sigma_0 \mathfrak{b}_l \subset \mathfrak{b}_{l+1}$  for all  $l$ . Then we have  $\mathfrak{b}_l = (\mathfrak{b}_l \cap F[X, \Delta^{-1}])_{\Sigma_l}$ .

*Proof.* We prove only (2). Then (1) can be proved by the same argument. Set  $\mathfrak{a} := \mathfrak{b}_l \cap F[X, \Delta^{-1}]$ , which is independent of  $l$  by Lemma 4.7. Suppose that  $\mathfrak{a}_{\Sigma_{l_0}} \subsetneq \mathfrak{b}_{l_0}$  for some  $l_0$ . Let  $(g_i)_{i \in I}$  be an  $F$ -basis of  $F[X, \Delta^{-1}]$  such that  $I = I_1 \amalg I_a$  and  $\mathfrak{a} = \oplus_{i \in I_a} Fg_i$ . Then we have

$$\Sigma_l[X, \Delta^{-1}] = (\oplus_{i \in I_1} \Sigma_l g_i) \oplus (\oplus_{i \in I_a} \Sigma_l g_i) = (\oplus_{i \in I_1} \Sigma_l g_i) \oplus \mathfrak{a}_{\Sigma_l}.$$

Since  $\mathfrak{a}_{\Sigma_{l_0}} \subsetneq \mathfrak{b}_{l_0}$ , we can take a minimal finite set  $J \subset I_1$  so that  $\mathfrak{b}_{l_0} \cap (\oplus_{i \in J} \Sigma_{l_0} g_i) \neq 0$ . By the injectivity of  $\sigma_0$  and the inclusion  $\sigma_0(\mathfrak{b}_l \cap \oplus_{i \in J} \Sigma_l g_i) \subset \mathfrak{b}_{l+1} \cap \oplus_{i \in J} \Sigma_{l+1} g_i$ ,  $J$  has the same properties for all  $l$ . We fix  $j \in J$  and consider the ideal of  $\Sigma_l \cong E[X, \Delta^{-1}]/\mathfrak{p}_l$ :

$$\mathfrak{m}_l := \{b \in \Sigma_l \mid \text{there exists } \sum_{i \in J} b_i g_i \in \mathfrak{b}_l \cap (\oplus_{i \in J} \Sigma_l g_i) \text{ such that } b_j = b\}.$$

Then  $\mathfrak{m}_l$  is a non-zero ideal by the minimality of  $J$ , and it is clear that  $\sigma \mathfrak{m}_l \subset \mathfrak{m}_{l+1}$ . By Lemma 4.4 we have  $\sigma \nu_l = \nu_{l+1} \sigma_1|_{E[X, \Delta^{-1}]}$ . Hence we can apply Lemma 4.9 to the inverse image of  $(\mathfrak{m}_l)_l$  in  $E[X, \Delta^{-1}]$ . Therefore we have  $\mathfrak{m}_l = \Sigma_l$ . Thus for each  $l$ , there exists an element  $h_l = \sum_{i \in J} b_{li} g_i \in \mathfrak{b}_l \cap (\oplus_{i \in J} \Sigma_l g_i)$  such that  $b_{lj} = 1$ . Then we have  $\mathfrak{b}_{l+1} \ni \sigma_0 h_l - h_{l+1} = \sum_{i \in J \setminus \{j\}} (\sigma b_{li} - b_{l+1, i}) g_i$ . By the minimality of  $J$ ,  $\sigma b_{li} = b_{l+1, i}$  for all  $i$  and  $l$ . We put  $b_i := (b_{li})_l \in \prod_l \Sigma_l$ . Then we have  $\sigma b_i = (\sigma b_{l-1, i})_l = (b_{li})_l = b_i$ . Hence  $b_i \in F$  and  $b_{li} \in F$  via the  $l$ -th projection. Then  $0 \neq h_l = \sum_{i \in J} b_{li} g_i \in \mathfrak{b}_l \cap F[X, \Delta^{-1}] = \mathfrak{a} = \oplus_{i \in I_a} Fg_i$ . This contradicts  $J \cap I_a = \emptyset$ .  $\square$

**Proposition 4.11.** (1) Let  $\psi : Z \times_E Z \rightarrow Z \times_E \mathrm{GL}_{r/E}$  be the morphism of affine  $E$ -schemes defined by  $(u, v) \mapsto (u, u^{-1}v)$  for any  $E$ -algebra  $S$  and any  $S$ -valued point  $(u, v) \in Z(S) \times Z(S)$ . Then  $\psi$  factors through an isomorphism  $\psi' : Z \times_E Z \rightarrow Z \times_E \Gamma_E$  of affine  $E$ -schemes.

(2) For any  $l$  and  $m$ , let  $\psi_{lm} : Z_l \times_E Z_{l+m} \rightarrow Z_l \times_E \mathrm{GL}_{r/E}$  be the morphism of affine  $E$ -schemes defined by  $(u, v) \mapsto (u, u^{-1}v)$  for any  $E$ -algebra  $S$  and any  $S$ -valued point  $(u, v) \in Z_l(S) \times Z_{l+m}(S)$ . Then  $\psi$  factors through an isomorphism  $\psi'_{lm} : Z_l \times_E Z_{l+m} \rightarrow Z_l \times_E \Gamma_{m,E}$  of affine  $E$ -schemes.

$$\begin{array}{ccc} Z \times Z & \xrightarrow{\psi; (u,v) \mapsto (u, u^{-1}v)} & Z \times \mathrm{GL}_{r/E} \\ & \searrow \psi' & \nearrow \text{natural} \\ & Z \times \Gamma_E & \end{array} \quad \begin{array}{ccc} Z_l \times Z_{l+m} & \xrightarrow{\psi_{lm}; (u,v) \mapsto (u, u^{-1}v)} & Z_l \times \mathrm{GL}_{r/E} \\ & \searrow \psi'_{lm} & \nearrow \text{natural} \\ & Z_l \times \Gamma_{m,E} & \end{array}$$

*Proof.* We prove only (2). Then (1) can be proved by the same argument. Let  $\alpha_l$  and  $\bar{\alpha}_{lm}$  be the homomorphisms (4.1) and (4.2). We restrict the domain of  $\bar{\alpha}_{lm}$  to  $\Sigma_l[X, \Delta^{-1}] \cong E[X, \Delta^{-1}]/\mathfrak{p}_l \otimes_E E[X, \Delta^{-1}]$  and the target of  $\bar{\alpha}_{lm}$  to  $\Sigma_l \otimes_E E[X, \Delta^{-1}]/\mathfrak{p}_m \cong E[X, \Delta^{-1}]/\mathfrak{p}_l \otimes_E E[X, \Delta^{-1}]/\mathfrak{p}_m$ . Then  $\psi_{lm}$  corresponds to  $\bar{\alpha}_{l, l+m}$  on the level of coordinate rings. Hence it is enough to show that  $\alpha_l^{-1} \mathfrak{p}_{m+l, \Sigma_l} = \mathfrak{q}_{m, \Sigma_l}$ .

Since we have an inclusion  $\sigma_0 \alpha_l^{-1} \mathfrak{p}_{m+l, \Sigma_l} = \alpha_{l+1}^{-1} \sigma_1 \mathfrak{p}_{m+l, \Sigma_l} \subset \alpha_{l+1}^{-1} \mathfrak{p}_{m+l+1, \Sigma_{l+1}}$ , the ideal  $\mathfrak{a}_m := \alpha_l^{-1} \mathfrak{p}_{m+l, \Sigma_l} \cap F[X, \Delta^{-1}]$  is independent of  $l$  by Lemma 4.7, and we can apply Lemma 4.10 to  $(\alpha_l^{-1} \mathfrak{p}_{m+l, \Sigma_l})_l$ . Then we have  $\alpha_l^{-1} \mathfrak{p}_{m+l, \Sigma_l} = (\alpha_l^{-1} \mathfrak{p}_{m+l, \Sigma_l} \cap F[X, \Delta^{-1}])_{\Sigma_l} =$

$\mathfrak{a}_{m,\Sigma_l}$ . On the other hand we have  $\alpha_l^{-1}\mathfrak{p}_{m+l,L_l} = \mathfrak{q}_{m,L_l}$  by Proposition 4.8. Therefore  $\mathfrak{q}_{m,L_l} = \alpha_l^{-1}\mathfrak{p}_{m+l,L_l} = \alpha_l^{-1}((\mathfrak{p}_{m+l,\Sigma_l})_{L_l}) = (\alpha_l^{-1}\mathfrak{p}_{m+l,\Sigma_l})_{L_l} = (\mathfrak{a}_{m,\Sigma_l})_{L_l} = \mathfrak{a}_{m,L_l}$ . Then we have  $\mathfrak{q}_m = \mathfrak{a}_m$  since  $L_l[X, \Delta^{-1}]$  is faithfully flat over  $F[X, \Delta^{-1}]$ . Thus we obtain  $\alpha_l^{-1}\mathfrak{p}_{m+l,\Sigma_l} = \mathfrak{a}_{m,\Sigma_l} = \mathfrak{q}_{m,\Sigma_l}$ .  $\square$

The next lemma is proved by an elementary argument. Thus we omit the proof. This lemma is applied to the  $S$ -valued points of the diagrams in Proposition 4.11 where  $S$  is an  $\bar{E}$ -algebra.

**Lemma 4.12.** (1) *Let  $G$  be a group,  $A$  and  $B$  be non-empty subsets of  $G$  such that the map*

$$\psi : A \times A \rightarrow A \times G; (u, v) \mapsto (u, u^{-1}v)$$

*factors through a bijection  $\psi' : A \times A \rightarrow A \times B$ . Then  $B$  is a subgroup of  $G$ ,  $A$  is stable under right-multiplication by elements of  $B$  and  $A$  becomes a  $B$ -torsor.*

(2) *Let  $G$  be a group,  $A_l$  and  $B_m$  be non-empty subsets of  $G$  such that the map*

$$\psi_{lm} : A_l \times A_{l+m} \rightarrow A_l \times G; (u, v) \mapsto (u, u^{-1}v)$$

*factors through a bijection  $\psi'_{lm} : A_l \times A_{l+m} \rightarrow A_l \times B_m$  for each  $l$  and  $m$ . Then  $B_0$  is a subgroup of  $G$ ,  $A_l$  is stable under right-multiplication by elements of  $B_0$  and  $A_l$  becomes a  $B_0$ -torsor for each  $l$ . Moreover, for any  $u \in A_l$  and  $v \in A_{l+m}$  there exists an element  $y \in B_m$  such that  $v = uy$ . The multiplication in  $G$  induces maps  $A_l \times B_m \rightarrow A_{l+m}$  and  $B_m \times B_{m'} \rightarrow B_{m+m'}$ , and the inversion in  $G$  induces a map  $B_m \rightarrow B_{-m}$ .*

By Proposition 4.11 and Lemma 4.12, we have surjective maps

$$(4.3) \quad Z_l(S) \times Z_{l+m}(S) \rightarrow \Gamma_m(S); (u, v) \mapsto u^{-1}v,$$

$$(4.4) \quad Z_l(S) \times \Gamma_m(S) \rightarrow Z_{l+m}(S); (x, y) \mapsto xy$$

for any  $\bar{E}$ -algebra  $S$ .

**Theorem 4.13.** (1) *The  $F$ -scheme  $\Gamma$  is a closed  $F$ -subgroup scheme of  $\mathrm{GL}_{r/F}$ , the  $E$ -scheme  $Z$  is stable under right multiplication by  $\Gamma_E$  and is a  $\Gamma_E$ -torsor.*

(2) *The  $F$ -scheme  $\Gamma_0$  is a closed  $F$ -subgroup schemes of  $\mathrm{GL}_{r/F}$ , the  $E$ -scheme  $Z_l$  is stable under right multiplication by  $\Gamma_{0,E}$  and is a  $\Gamma_{0,E}$ -torsor for each  $l$ .*

(3) *The  $F$ -scheme  $\Gamma_m$  is stable under right and left multiplications by  $\Gamma_0$  and is a  $\Gamma_0$ -torsor for each  $m$ .*

*Proof.* We prove only (2). Then (1) and (3) can be proved by the same argument. By Proposition 4.11, we have a bijection  $Z_l(S) \times Z_{l+m}(S) \rightarrow Z_l(S) \times \Gamma_m(S); (u, v) \mapsto (u, u^{-1}v)$  for any  $\bar{E}$ -algebra  $S$ . Since  $Z_l(S)$  is non-empty, Lemma 4.12 implies that  $\Gamma_{0,\bar{E}}$  is a closed subgroup scheme of  $\mathrm{GL}_{r/\bar{E}}$  and  $Z_{l,\bar{E}}$  is a  $\Gamma_{0,\bar{E}}$ -torsor. Therefore  $\Gamma_0$  is a closed subgroup scheme of  $\mathrm{GL}_{r/F}$  by the faithfully flatness of the inclusion  $F \rightarrow \bar{E}$ . Similarly,  $Z_l$  is a  $\Gamma_{0,E}$ -torsor by the faithfully flatness of the inclusion  $E \rightarrow \bar{E}$ .  $\square$

**Theorem 4.14.** (a) *The  $E$ -schemes  $Z$  and  $Z_l$  are smooth.*

(b) *The  $F$ -schemes  $\Gamma$  and  $\Gamma_m$  are smooth.*

(c) *If  $E$  is algebraically closed in the fraction field  $\Lambda_{l_0}$  of  $\Sigma_{l_0}$  for some  $l_0$ , then  $Z_l$  and  $\Gamma_m$  are absolutely irreducible.*

(d)  $\dim \Gamma = \dim \Gamma_m = \mathrm{tr.deg}_E \Lambda_l$ .

*Proof.* (a), (b) Since  $L_l/E$  is a separable extension,  $\Lambda/E$  is also a separable extension, where  $\Lambda = \text{Frac}(\Sigma)$  the total ring of fractions of  $\Sigma$ . Thus for any field extension  $\Omega/E$ ,  $\Lambda \otimes_E \Omega$  is reduced. Therefore  $\Sigma \otimes_E \Omega$  is reduced and  $Z = \text{Spec } \Sigma$  is absolutely reduced. Since  $\Gamma_{\bar{E}} \cong Z_{\bar{E}}$ ,  $\Gamma$  is absolutely reduced. Since  $\Gamma$  is an algebraic group, the property that  $\Gamma$  is absolutely reduced implies that  $\Gamma$  is smooth. Again since  $\Gamma_{\bar{E}} \cong Z_{\bar{E}}$ , we have that  $Z$  is smooth. The statements of  $Z_l$  and  $\Gamma_m$  are proved similarly.

(c) For any field extension  $\Omega/E$ ,  $\Lambda_{l_0} \otimes_E \Omega$  is an integral domain by the assumption. Therefore  $\Sigma_{l_0} \otimes_E \Omega$  is an integral domain and  $Z_{l_0}$  is absolutely integral. Since  $Z_{l,\bar{E}} \cong \Gamma_{m,\bar{E}}$  for all  $l$  and  $m$ ,  $Z_l$  and  $\Gamma_m$  are all absolutely integral.

(d) We have an equality  $\dim \Gamma = \dim \Gamma_m = \dim \Gamma_0 = \dim Z_l = \text{tr.deg}_E \Lambda_l$ .  $\square$

**Corollary 4.15.** (1) *There exists a divisor  $d'$  of  $d$  such that if  $l \equiv l' \pmod{d'}$  then  $Z_l = Z_{l'}$  and if  $l \not\equiv l' \pmod{d'}$  then  $Z_l \cap Z_{l'} = \emptyset$ .*

(2) *If  $m \equiv m' \pmod{d'}$  then  $\Gamma_m = \Gamma_{m'}$  and if  $m \not\equiv m' \pmod{d'}$  then  $\Gamma_m \cap \Gamma_{m'} = \emptyset$ .*

Therefore we can write  $Z = \coprod_{l \in \mathbb{Z}/d'} Z_l$ ,  $\Sigma = \prod_{l \in \mathbb{Z}/d'} \Sigma_l$ ,  $\Lambda = \prod_{l \in \mathbb{Z}/d'} \Lambda_l$  and  $\Gamma = \prod_{m \in \mathbb{Z}/d'} \Gamma_m$ .

*Proof.* (1) Since  $Z_l$  is a  $\Gamma_{0,E}$ -torsor and absolutely reduced for all  $l$ , it is clear that  $Z_l = Z_{l'}$  or  $Z_l \cap Z_{l'} = \emptyset$ . We have the surjective map (4.4) :  $Z_l(\bar{E}) \times \Gamma_1(\bar{E}) \rightarrow Z_{l+1}(\bar{E})$ . Therefore if  $Z_l = Z_{l'}$ , then  $Z_{l+1}(\bar{E}) = Z_{l'+1}(\bar{E})$ . Hence if we take  $d'$  to be the minimum positive integer such that  $Z_0 = Z_{d'}$ , then  $d'$  satisfies the desired properties.

(2) By the same argument of the proof of (1), there exists a divisor  $d''$  of  $d$  which is the period of  $(\Gamma_m)_m$ . Then by the map (4.3), we have

$$\Gamma_m(\bar{E}) = Z_l(\bar{E})^{-1} Z_{l+m}(\bar{E}) = Z_l(\bar{E})^{-1} Z_{l+m+d'}(\bar{E}) = \Gamma_{m+d'}(\bar{E}).$$

This means that  $d''|d'$ . By the map (4.4), we have

$$Z_l(\bar{E}) = Z_l(\bar{E})\Gamma_0(\bar{E}) = Z_l(\bar{E})\Gamma_{d''}(\bar{E}) = Z_{l+d''}(\bar{E}).$$

This means that  $d'|d''$ .  $\square$

## 4.2 $\Gamma$ -action

For any  $F$ -algebras  $R$  and  $S$ , we set  $S^{(R)} := R \otimes_F S$ . In particular, if  $R = F'$  is a field, we set  $S' := S^{(F')}$ . If  $\sigma$  acts on  $S$ , we define the  $\sigma$ -action on  $S^{(R)}$  by  $\text{id} \otimes \sigma$ . Note that, if  $S^\sigma = F$ , then we have  $(S^{(R)})^\sigma = R$ . Let  $\text{Aut}_\sigma(\Sigma^{(R)}/E^{(R)})$  denote the group of automorphisms of  $\Sigma^{(R)}$  over  $E^{(R)}$  that commute with  $\sigma$ . Similarly we define  $\text{Aut}_\sigma(\Lambda^{(R)}/E^{(R)})$ . For any  $\gamma \in \Gamma(R)$ , we obtain an automorphism  $Z_{E^{(R)}} \rightarrow Z_{E^{(R)}}; x \mapsto x\gamma$ . On the level of coordinate rings, this corresponds to an automorphism  $\Sigma^{(R)} \rightarrow \Sigma^{(R)}; h(\Psi) \mapsto \gamma.h(\Psi) := h(\Psi\gamma)$ . Note that  $\Sigma^{(R)} = E^{(R)} \otimes_E \Sigma = E^{(R)}[\Psi, \Delta(\Psi)^{-1}] \cong E^{(R)}[X, \Delta^{-1}]/\mathfrak{p}_{E^{(R)}}$ . Thus we have a group homomorphism  $\kappa_R : \Gamma(R) \rightarrow \text{Aut}(\Sigma^{(R)}/E^{(R)})$ .

**Lemma 4.16.** (1) *For any  $F$ -algebra  $R$ , the map  $\kappa_R$  induces an isomorphism  $\Gamma(R) \xrightarrow{\sim} \text{Aut}_\sigma(\Sigma^{(R)}/E^{(R)})$ . Its inverse is the map  $\alpha \mapsto \Psi^{-1}(\alpha\Psi_{ij})_{ij}$ .*

(2)  $\text{Aut}_\sigma(\Sigma/E) \cong \text{Aut}_\sigma(\Lambda/E)$ .

(3) *If  $\Lambda_l/F$  is a regular extension (i.e. separable extension and  $F$  is algebraically closed in  $\Lambda_l$ ) for all  $l$  and  $F'/F$  is an algebraic extension of fields, then we have  $\text{Aut}_\sigma(\Sigma'/E') \cong \text{Aut}_\sigma(\Lambda'/E')$ .*

*Proof.* (1) For any  $\gamma \in \Gamma(R)$  and  $h(\Psi) \in \Sigma^{(R)}$ , we have  $\sigma(\gamma.h(\Psi)) = \sigma(h(\Psi\gamma)) = h^\sigma(\sigma(\Psi\gamma)) = h^\sigma(\Phi\Psi\gamma) = \gamma.(h^\sigma(\Phi\Psi)) = \gamma.(h^\sigma(\sigma\Psi)) = \gamma.(\sigma(h(\Psi)))$ . Hence  $\kappa_R(\gamma)$  commutes with  $\sigma$ . Suppose that  $\kappa_R(\gamma)$  is the identity. Then  $h(\Psi\gamma) = h(\Psi)$  for any  $h(\Psi) \in \Sigma^{(R)}$ . In particular if we take  $h(\Psi) = \Psi_{ij}$  for each  $i$  and  $j$ , then we obtain  $\Psi\gamma = \Psi$  in  $\text{GL}_r(\Sigma^{(R)})$ . Therefore  $\gamma = 1$  and this means that  $\kappa_R$  is injective. Conversely, let  $\alpha \in \text{Aut}_\sigma(\Sigma^{(R)}/E^{(R)})$  be any element. Then  $\alpha$  corresponds to an automorphism  $\bar{\alpha} : Z_{E^{(R)}} \rightarrow Z_{E^{(R)}}$ , and  $\bar{\alpha}$  maps the  $\Sigma^{(R)}$ -valued point  $\Psi$  to  $(\alpha\Psi_{ij})_{ij}$ . By Theorem 4.13, there exists an element  $\gamma \in \Gamma(\Sigma^{(R)})$  such that  $\Psi\gamma = (\alpha\Psi_{ij})_{ij}$ . Then for any  $h(\Psi) \in \Sigma^{(R)}$ , we have  $\alpha(h(\Psi)) = h((\alpha\Psi_{ij})_{ij}) = h(\Psi\gamma)$ . Thus we obtain  $\sigma(\gamma.h(\Psi)) = \sigma(\alpha(h(\Psi))) = \alpha(\sigma(h(\Psi))) = \alpha(h^\sigma(\Phi\Psi)) = \gamma.h^\sigma(\Phi\Psi) = h^\sigma(\Phi\Psi\gamma)$ . On the other hand,  $\sigma(\gamma.h(\Psi)) = \sigma(h(\Psi\gamma)) = h^\sigma((\sigma\Psi)(\sigma\gamma)) = h^\sigma(\Phi\Psi(\sigma\gamma))$ . If we take  $h(\Psi) = \Psi_{ij}$  for each  $i$  and  $j$ , we obtain  $\Phi\Psi(\sigma\gamma) = \Phi\Psi\gamma$ . Hence  $\sigma\gamma = \gamma$ . Therefore we have  $\gamma \in \Gamma(R)$  and  $\kappa_R(\gamma) = \alpha$ .

(2) Since  $\Lambda = \text{Frac}(\Sigma)$ , any automorphism of  $\Sigma$  extends uniquely to an automorphism of  $\Lambda$ . Conversely if  $\alpha \in \text{Aut}_\sigma(\Lambda/E)$ , then  $\sigma(\alpha(\Psi)) = \alpha(\sigma\Psi) = \alpha(\Phi\Psi) = \Phi \cdot \alpha(\Psi)$  in  $\text{GL}_r(L)$ . Thus we have  $\alpha(\Psi) \in \Psi \cdot \text{GL}_r(F)$ . This implies that  $\alpha(\Sigma) \subset \Sigma$ . Similarly, we have  $\alpha^{-1}(\Sigma) \subset \Sigma$ . Therefore  $\alpha(\Sigma) = \Sigma$ .

(3) Since  $\Lambda_l/F$  is a regular extension, so is  $E/F$ . Since  $F'/F$  is an algebraic extension,  $E'$  and  $\Lambda'_l$  are fields and  $\Lambda'_l = \text{Frac}(\Sigma'_l)$ . Therefore  $\Lambda' = \prod_{l \in \mathbb{Z}/d'} \Lambda'_l$  is a finite product of fields and  $\Lambda' = \text{Frac}(\Sigma')$ . Then, the proof is the same as (2).  $\square$

We prepare some lemmas about Zariski density.

**Lemma 4.17.** *Let  $\Omega/k$  be a field extension such that  $\Omega$  is an algebraically closed field,  $X$  an algebraic variety over  $k$  and  $Y$  a closed subvariety of  $X_\Omega$ . If  $X(k) \cap Y(\Omega)$  is Zariski dense in  $Y$ , then  $Y$  is defined over  $k$ , i.e. there exists some algebraic variety  $Y_0$  over  $k$  such that  $Y = Y_{0,\Omega}$ .*

*Proof.* We may assume that  $X$  is affine. Let  $k[X]$  be the coordinate ring of  $X$  and  $\Omega[X]$  the coordinate ring of  $X_\Omega$ . Then we have  $\Omega[X] = \Omega \otimes_k k[X]$ . Let  $\mathfrak{a} \subset \Omega[X]$  be the defining ideal of  $Y$ ,  $\mathfrak{a}_k := \mathfrak{a} \cap k[X]$  and  $\mathfrak{a}_{k,\Omega} := \mathfrak{a}_k \cdot \Omega[X]$ . We need to show that  $\mathfrak{a}_{k,\Omega} = \mathfrak{a}$ . Thus we assume that  $\mathfrak{a}_{k,\Omega} \subsetneq \mathfrak{a}$ . Let  $(g_i)_{i \in I'}$  be a  $k$ -basis of  $k[X]$  such that  $(g_i)_{i \in I}$  is a  $k$ -basis of  $\mathfrak{a}_k$  for some  $I \subset I'$ . We also take  $(c_j)_{j \in J}$  to be a  $k$ -basis of  $\Omega$ . Since  $\mathfrak{a}_{k,\Omega} \subsetneq \mathfrak{a}$ , there exists a non-zero element  $f = \sum_{i \in I' \setminus I} a_i g_i \in \mathfrak{a}$  where  $a_i \in \Omega$ . Write  $a_i = \sum_j \alpha_{ij} c_j$  ( $\alpha_{ij} \in k$ ). Then we can write  $f = \sum_j c_j \sum_{i \in I' \setminus I} \alpha_{ij} g_i$ . For any  $x \in X(k) \cap Y(\Omega)$ , we have  $\sum_j c_j \sum_{i \in I' \setminus I} \alpha_{ij} g_i(x) = f(x) = 0$ . Since  $(c_j)_j$  is linearly independent over  $k$ , we have  $\sum_{i \in I' \setminus I} \alpha_{ij} g_i(x) = 0$  for all  $j$ . By the density assumption, we obtain  $\sum_{i \in I' \setminus I} \alpha_{ij} g_i(x) = 0$  for all  $x \in Y(\Omega)$ . Therefore  $\sum_{i \in I' \setminus I} \alpha_{ij} g_i \in \mathfrak{a} \cap k[X] = \mathfrak{a}_k = \oplus_{i \in I} k g_i$  for all  $j$ . Since  $(g_i)_{i \in I'}$  is linearly independent over  $k$ , we have  $\alpha_{ij} = 0$ . Thus  $f = 0$ , which is a contradiction.  $\square$

**Corollary 4.18.** *Let  $\Omega/k$  be a field extension such that  $\Omega$  is an algebraically closed field and  $X$  an algebraic variety over  $k$ . If  $X(k)$  is Zariski dense in  $X$ , then  $X(k)$  is Zariski dense in  $X_\Omega$ .*

*Proof.* We take  $Y$  to be the Zariski closure of  $X(k)$  in  $X_\Omega$ . Then there exists some algebraic variety  $Y_0$  over  $k$  such that  $Y = Y_{0,\Omega}$  by Lemma 4.17. It is clear that  $X(k) \subset Y_0(k)$ . Hence we have  $Y_0 = X$  since  $X(k)$  is Zariski dense in  $X$ . Therefore we have  $Y = X_\Omega$ .  $\square$

**Lemma 4.19.** *Let  $\Omega/k$  be a field extension,  $X_1$  an algebraic variety over  $\Omega$  and  $X_2$  an algebraic varieties over  $k$ . If  $X_2(k)$  is Zariski dense in  $X_2$ , then  $X_1(\Omega) \times X_2(k)$  is Zariski dense in  $X_1 \times_{\Omega} X_{2,\Omega}$ , where  $\bar{\Omega}$  is an algebraic closure of  $\Omega$ .*

*Proof.* Let  $V$  be the Zariski closure of  $X_1(\bar{\Omega}) \times X_2(k)$  in  $X_1 \times_{\Omega} X_{2,\Omega}$ . We assume that  $V \subsetneq X_1 \times_{\Omega} X_{2,\Omega}$ . Then we have an element  $(x, y) \in (X_1(\bar{\Omega}) \times X_2(\bar{\Omega})) \setminus V(\bar{\Omega})$ . Therefore we have  $(\{x\} \times X_{2,\bar{\Omega}}) \cap V_{\bar{\Omega}} \subsetneq \{x\} \times X_{2,\bar{\Omega}}$  and  $\{x\} \times X_2(k) \subset (\{x\} \times X_2(\bar{\Omega})) \cap V(\bar{\Omega})$ . On the other hand, since  $X_2(k)$  is Zariski dense in  $X_2$ ,  $X_2(k)$  is Zariski dense in  $X_{2,\bar{\Omega}}$  by Corollary 4.18. Then  $\{x\} \times X_2(k)$  is Zariski dense in  $\{x\} \times_{\bar{\Omega}} X_{2,\bar{\Omega}}$ . This is a contradiction.  $\square$

**Theorem 4.20.** *Let  $F'/F$  be an algebraic extension of fields such that  $\Gamma(F')$  is Zariski dense in  $\Gamma_{F'}$ . Assume that  $F' = F$  or  $\Lambda_l/F$  is a regular extension for all  $l$ . Then we have  $(\Lambda')^{\Gamma(F')} = E'$  and  $\Lambda \cap (\Lambda')^{\Gamma(F')} = E$ .*

*Proof.* The second part follows from the first part and the assumptions. Thus we prove the first part. In the proof of this theorem, we regard  $l$  as an element of the index set  $\mathbb{Z}/d'$ . We take any element  $f = (f_l)_l \in (\Lambda')^{\Gamma(F')} \subset \prod_{l \in \mathbb{Z}/d'} \Lambda'_l$ , and consider  $f_l$  as a rational function of  $Z_{l,E'}$  to  $\mathbb{A}_{E'}^1$ . Then, for some non-empty open affine set  $U_l \subset Z_{l,E'}$ ,  $f_l$  can be regarded as a morphism  $f_l : U_l \rightarrow \mathbb{A}_{E'}^1$ . By Proposition 4.11, we have an isomorphism  $Z_{E'} \times_{E'} \Gamma_{E'} \rightarrow Z_{E'} \times_{E'} Z_{E'}; (x, y) \mapsto (x, xy)$ . We set  $U \subset Z_{E'} \times_{E'} \Gamma_{E'}$  to be the open subset corresponding to  $\coprod_l U_l \times_{E'} \coprod_l U_l$  via this isomorphism, and consider the two maps

$$g_i : U \longrightarrow \coprod U_l \times_{E'} \coprod U_l \xrightarrow{\pi_i} \coprod U_l \xrightarrow{f=(f_l)} \mathbb{A}_{E'}^1$$

where  $i = 1, 2$  and  $\pi_i$  is the  $i$ -th projection. Let  $S$  be an algebraic closure of  $E'$ . Then for any  $(x, y) \in (Z(S) \times \Gamma(F')) \cap U(S)$ , we have  $g_1(x, y) = f(\pi_1(x, xy)) = f(x)$  and  $g_2(x, y) = f(\pi_2(x, xy)) = f(xy) = f(x)$  since  $f$  is fixed by  $\Gamma(F')$ . Since  $\Gamma(F')$  is Zariski dense in  $\Gamma_{F'}$ ,  $Z(S) \times \Gamma(F')$  is Zariski dense in  $Z_{E'} \times_{E'} \Gamma_{E'}$  by Lemma 4.19. Then  $(Z(S) \times \Gamma(F')) \cap U(S)$  is Zariski dense in  $U$ . Thus we have  $g_1 = g_2$ , and this means  $f\pi_1 = f\pi_2$ . By considering on the level of coordinate rings, it is clear that  $f \in E'$  since  $E'$  is a field.  $\square$

**Corollary 4.21.** *If  $F$  is a local field, and each connected component of  $\Gamma$  has an  $F$ -valued point, then  $\Lambda^{\Gamma(F)} = E$ .*

*Proof.* Take any connected component  $\Gamma'$  of  $\Gamma$ . Then there exists an  $F$ -valued point  $x \in \Gamma'(F)$  by the assumption, and  $\Gamma'$  is smooth by Lemma 4.14. By the implicit function theorem, there exists an open neighborhood of  $x$  in  $\Gamma'(F)$  which is isomorphic to some open subset of  $F^{\dim \Gamma}$ . Since  $\Gamma'$  is irreducible, this implies that  $\Gamma'(F)$  is Zariski dense in  $\Gamma'$ . Hence we conclude that  $\Gamma(F)$  is Zariski dense in  $\Gamma$ . Then this corollary follows from Theorem 4.20.  $\square$

## 5 The group $\Gamma$ and $\varphi$ -modules

### 5.1 General case

In this subsection, we use the notations defined in Section 3, and fix a  $\sigma$ -admissible triple  $(F, E, L)$ . Let  $M \in \Phi M_E^L$  be an  $L$ -trivial  $\varphi$ -module over  $E$  of rank  $r$ ,  $\mathcal{T}_M$  the Tannakian subcategory of  $\Phi M_E^L$  generated by  $M$ ,  $V_M : \mathcal{T}_M \rightarrow \mathbf{Vec}(F)$  the fiber functor of  $\mathcal{T}_M$  and  $\Gamma_M$  the Tannakian Galois group of  $(\mathcal{T}_M, V_M)$ . We fix  $\mathbf{m} \in \text{Mat}_{r \times 1}(M)$  an  $E$ -basis of  $M$ . Then

there exist matrices  $\Phi \in \mathrm{GL}_r(E)$  and  $\Psi \in \mathrm{GL}_r(L)$  such that  $\varphi \mathbf{m} = \Phi \mathbf{m}$  and  $\sigma \Psi = \Phi \Psi$ . We define  $\Gamma, \Sigma, \dots$  as in Section 4 for  $\Phi$  and  $\Psi$ . In this subsection, we show that there exists an equivalence of categories  $\mathcal{T}_M \xrightarrow{\sim} \mathbf{Rep}(\Gamma, F)$  under some assumptions. Note that  $\Sigma$  and  $\Sigma_l$  are independent of the choice of  $\mathbf{m}$  and  $\Psi$  by Proposition 3.13. If  $N \in \mathcal{T}_M$  and  $s$  is the rank of  $N$  over  $E$ , we use the notation  $\mathbf{n} \in \mathrm{Mat}_{s \times 1}(N)$  and  $\Psi_N \in \mathrm{GL}_s(L)$  for an  $E$ -basis of  $N$  and a fundamental matrix respectively. For any  $F$ -algebras  $S$  and  $R$ , we set  $S^{(R)} := R \otimes_F S$ .

**Proposition 5.1.** *For any  $N \in \mathcal{T}_M$ , we have  $\Psi_N \in \mathrm{GL}_s(\Sigma)$ .*

*Proof.* Let  $N$  and  $N'$  be objects in  $\mathcal{T}_M$ . Set  $s := \dim_E N$  and  $s' := \dim_E N'$  and assume that  $\Psi_N \in \mathrm{GL}_s(\Sigma)$  and  $\Psi_{N'} \in \mathrm{GL}_{s'}(\Sigma)$ . Since we can take  $\Psi_{N \oplus N'} = \Psi_N \oplus \Psi_{N'}$ ,  $\Psi_{N \otimes N'} = \Psi_N \otimes \Psi_{N'}$  and  $\Psi_{N^\vee} = (\Psi_N^{-1})^{\mathrm{tr}}$ , we have that  $\Psi_{N \oplus N'}$ ,  $\Psi_{N \otimes N'}$  and  $\Psi_{N^\vee}$  are invertible matrices with coefficients in  $\Sigma$ . We have to show that if  $0 \rightarrow N' \rightarrow N \rightarrow N'' \rightarrow 0$  is an exact sequence in  $\mathcal{T}_M$  and  $\Psi_N \in \mathrm{GL}_s(\Sigma)$ , then  $\Psi_{N'} \in \mathrm{GL}_{s'}(\Sigma)$  and  $\Psi_{N''} \in \mathrm{GL}_{s''}(\Sigma)$ . Let  $\mathbf{n}, \mathbf{n}'$  and  $\mathbf{n}''$  be  $E$ -bases of  $N, N'$  and  $N''$  such that

$$\mathbf{n} = \begin{bmatrix} \mathbf{n}' \\ \tilde{\mathbf{n}}'' \end{bmatrix}$$

where  $\tilde{\mathbf{n}}''$  is a lift of  $\mathbf{n}''$ . Since  $V_M$  is exact, we have an exact sequence

$$0 \rightarrow V_M(N') \rightarrow V_M(N) \rightarrow V_M(N'') \rightarrow 0.$$

Let  $\mathbf{x}, \mathbf{x}'$  and  $\mathbf{x}''$  be  $F$ -bases of  $V_M(N), V_M(N')$  and  $V_M(N'')$  such that

$$\mathbf{x} = \begin{bmatrix} \mathbf{x}' \\ \tilde{\mathbf{x}}'' \end{bmatrix}$$

where  $\tilde{\mathbf{x}}''$  is a lift of  $\mathbf{x}''$ . By Proposition 3.14, there exist matrices  $A \in \mathrm{GL}_s(F)$ ,  $A' \in \mathrm{GL}_{s'}(F)$  and  $A'' \in \mathrm{GL}_{s''}(F)$  such that  $\Psi_N^{-1} \mathbf{n} = A \mathbf{x}$ ,  $\Psi_{N'}^{-1} \mathbf{n}' = A' \mathbf{x}'$  and  $\Psi_{N''}^{-1} \mathbf{n}'' = A'' \mathbf{x}''$ . Consider the exact sequence

$$0 \rightarrow L \otimes_E N' \rightarrow L \otimes_E N \rightarrow L \otimes_E N'' \rightarrow 0.$$

Since both  $\Psi_{N''} A'' \tilde{\mathbf{x}}''$  and  $\tilde{\mathbf{n}}''$  are mapped to  $\mathbf{n}''$  and  $\mathbf{x}'$  is an  $L$ -basis of  $L \otimes_E N'$ , there exists a matrix  $B \in \mathrm{Mat}_{s'' \times s'}(L)$  such that

$$\tilde{\mathbf{n}}'' = B \mathbf{x}' + \Psi_{N''} A'' \tilde{\mathbf{x}}''.$$

Therefore we have

$$\Psi_N A \mathbf{x} = \mathbf{n} = \begin{bmatrix} \mathbf{n}' \\ \tilde{\mathbf{n}}'' \end{bmatrix} = \begin{bmatrix} \Psi_{N'} A' & 0 \\ B & \Psi_{N''} A'' \end{bmatrix} \mathbf{x}.$$

Since  $\Psi_N \in \mathrm{GL}_s(\Sigma)$ , we conclude that  $\Psi_{N'} \in \mathrm{GL}_{s'}(\Sigma)$  and  $\Psi_{N''} \in \mathrm{GL}_{s''}(\Sigma)$ .  $\square$

**Lemma 5.2.** *For any  $N \in \mathcal{T}_M$  and  $F$ -algebra  $R$ , there exists a natural isomorphism*

$$\Sigma^{(R)} \otimes_F V(N) \rightarrow \Sigma^{(R)} \otimes_E N.$$

*Similarly, there exists a natural isomorphism*

$$\Sigma_l^{(R)} \otimes_F V(N) \rightarrow \Sigma_l^{(R)} \otimes_E N$$

*for all  $l$ .*

*Proof.* The inclusion  $V(N) \subset \Sigma \otimes_E N$  and the product map  $\Sigma \otimes_F \Sigma \rightarrow \Sigma$  induce a natural map

$$\kappa : \Sigma^{(R)} \otimes_F V(N) \hookrightarrow \Sigma^{(R)} \otimes_F \Sigma \otimes_E N \rightarrow \Sigma^{(R)} \otimes_E N.$$

Since  $1 \otimes \Psi_N^{-1} \mathbf{n}$  is a  $\Sigma^{(R)}$ -basis of  $\Sigma^{(R)} \otimes_F V(N)$ , we can write  $\kappa$  explicitly as follows:

$$\kappa(\mathbf{f} \cdot (1 \otimes \Psi_N^{-1} \mathbf{n})) = (\mathbf{f} \Psi_N^{-1}) \cdot (1 \otimes \mathbf{n})$$

for all  $\mathbf{f} \in \text{Mat}_{1 \times s}(\Sigma^{(R)})$ . Hence it is clear that  $\kappa$  is an isomorphism. The  $\Sigma_l$  version is proved by the same argument.  $\square$

**Theorem 5.3.** *For any  $N \in \mathcal{T}_M$ , there exists a natural representation*

$$\rho_N : \Gamma \rightarrow \text{GL}(V(N))$$

*over  $F$  that is functorial in  $N$ .*

*Proof.* For any  $F$ -algebra  $R$  and  $\gamma \in \Gamma(R) \subset \text{GL}_r(R)$ , we define

$$\rho_N^{(R)}(\gamma) : R \otimes_F V(N) \hookrightarrow \Sigma^{(R)} \otimes_F V(N) \rightarrow \Sigma^{(R)} \otimes_E N \rightarrow \Sigma^{(R)} \otimes_E N,$$

where the second map is the isomorphism defined in Lemma 5.2 and the third map is defined by  $h(\Psi) \otimes x \mapsto h(\Psi\gamma) \otimes x$ . Clearly  $\rho_N^{(R)}$  is functorial in  $N$ . If  $\text{im}(\rho_M^{(R)}(\gamma)) = R \otimes_F V(M)$  then  $\text{im}(\rho_N^{(R)}(\gamma)) = R \otimes_F V(N)$  for all  $N \in \mathcal{T}_M$ . Thus we may assume that  $N = M$ . We can write  $\rho_M^{(R)}(\gamma)$  explicitly:

$$\rho_M^{(R)}(\gamma)(\mathbf{f} \cdot (1 \otimes \Psi^{-1} \mathbf{m})) = \mathbf{f} \gamma^{-1} (1 \otimes \Psi^{-1} \mathbf{m}),$$

for each  $\mathbf{f} \in \text{Mat}_{1 \times r}(R)$ . Therefore we have  $\text{im}(\rho_M^{(R)}(\gamma)) = R \otimes_F V(M)$ .  $\square$

From the above description of  $\rho_M^{(R)}$ , we have the following corollary:

**Corollary 5.4.** *The representation  $\rho_M : \Gamma \rightarrow \text{GL}(V(M))$  is faithful.*

From Theorem 5.3, we have a functor  $\xi_M : \mathcal{T}_M \rightarrow \mathbf{Rep}(\Gamma, F)$ , and it is clear by the construction that  $\xi_M$  is a tensor functor. Let  $\eta_M : \mathbf{Rep}(\Gamma_M, F) \rightarrow \mathcal{T}_M$  be the equivalence of categories defined by the Tannakian duality and  $\alpha : \mathbf{Rep}(\Gamma, F) \rightarrow \mathbf{Vec}(F)$  the forgetful functor. Since  $V_M = \alpha \circ \xi_M$ , there exists a unique homomorphism  $\pi_M : \Gamma \rightarrow \Gamma_M$  over  $F$  such that the natural functor  $\tau_M : \mathbf{Rep}(\Gamma_M, F) \rightarrow \mathbf{Rep}(\Gamma, F)$  induced by  $\pi_M$  satisfies  $\xi_M \circ \eta_M = \tau_M$ .

$$\begin{array}{ccc} \mathbf{Rep}(\Gamma_M, F) & \xrightarrow{\eta_M} & \mathcal{T}_M \xrightarrow{\xi_M} \mathbf{Rep}(\Gamma, F) \\ & & \searrow V_M \quad \downarrow \alpha \\ & & \mathbf{Vec}(F) \end{array}$$

**Proposition 5.5.** *For any representation  $W \in \mathbf{Rep}(\Gamma, F)$ , there exists an object  $N \in \mathcal{T}_M$  such that  $W$  is isomorphic to a subquotient of  $\xi_M(N)$ .*

*Proof.* By Corollary 5.4, the  $\Gamma$ -representation  $\xi_M(M) = \rho_M$  is faithful. Therefore,  $W$  is isomorphic to a subquotient of representation of the form

$$\oplus_{i=1}^n (\xi_M(M))^{\otimes a_i} \otimes (\xi_M(M)^\vee)^{\otimes b_i},$$

where  $a_i, b_i \in \mathbb{N}$ . However we have  $\oplus_{i=1}^n (\xi_M(M))^{\otimes a_i} \otimes (\xi_M(M)^\vee)^{\otimes b_i} = \xi_M(\oplus_{i=1}^n M^{\otimes a_i} \otimes (M^\vee)^{\otimes b_i})$ .  $\square$

Proposition 5.5 is equivalent to the next theorem ([6], Proposition 2.21).

**Theorem 5.6.** *The morphism of affine  $F$ -schemes  $\pi_M : \Gamma \rightarrow \Gamma_M$  is a closed immersion.*

From now on, we assume that  $\Gamma(F)$  is Zariski dense in  $\Gamma$  or  $\Lambda_l/F$  is a regular extension for each  $l$ . In the former case we put  $F' = F$ , and in the latter case we put  $F' = \bar{F}$ . For any  $F$ -algebra  $S$ , we set  $S' := F' \otimes_F S$ . Then in any case,  $E'$  and  $\Lambda'_l$  are fields,  $\Lambda'_l = \text{Frac}(\Sigma'_l)$  and  $\Lambda \cap (\Lambda')^{\Gamma(F')} = E$  by Theorem 4.20.

**Proposition 5.7.** *Assume that  $\Gamma(F)$  is Zariski dense in  $\Gamma$  or  $\Lambda_l/F$  is a regular extension for each  $l$ . Then the functor  $\xi_M : \mathcal{T}_M \rightarrow \mathbf{Rep}(\Gamma, F)$  is fully faithful.*

*Proof.* For any objects  $N, N' \in \mathcal{T}_M$ , there exist natural isomorphisms  $\text{Hom}_{\mathcal{T}_M}(N', N) \cong \text{Hom}_{\mathcal{T}_M}(\mathbf{1}, \text{Hom}(N', N))$  and  $\text{Hom}_\Gamma(V(N'), V(N)) \cong \text{Hom}_\Gamma(V(\mathbf{1}), V(\text{Hom}(N', N)))$ . Thus it is enough to show that, for any  $N \in \mathcal{T}_M$ ,  $\text{Hom}_{\mathcal{T}_M}(\mathbf{1}, N) \rightarrow \text{Hom}_\Gamma(V(\mathbf{1}), V(N))$  is an isomorphism. It is injective since  $\text{Hom}_{\mathcal{T}_M}(\mathbf{1}, N) = N^\varphi = N \cap V(N) \hookrightarrow \text{Hom}_\Gamma(V(\mathbf{1}), V(N))$ . For any  $\phi \in \text{Hom}_\Gamma(V(\mathbf{1}), V(N))$ , there exists  $\mathbf{h} = \mathbf{h}(\Psi) \in \text{Mat}_{1 \times s}(\Sigma)$  so that  $\phi(1) = \mathbf{h}\mathbf{n}$  by Lemma 5.2. Then for any  $\gamma \in \Gamma(F')$ , we have  $\mathbf{h}(\Psi)\mathbf{n} = \phi(1) = \gamma \cdot \phi(1) = \mathbf{h}(\Psi\gamma)\mathbf{n}$ . Hence  $\mathbf{h}(\Psi) = \mathbf{h}(\Psi\gamma) = \gamma \cdot \mathbf{h}$ . By Theorem 4.20, we have  $\mathbf{h} \in \text{Mat}_{1 \times s}(E)$ , and this implies  $\phi(1) = \mathbf{h}\mathbf{n} \in N \cap V(N)$ .  $\square$

We prepare a lemma from linear algebra.

**Lemma 5.8.** *Let  $E \subset \Lambda$  be general rings where  $E$  is a field and  $\Lambda = \prod_{l \in \mathbb{Z}/d'} \Lambda_l$  is a finite product of fields. Assume that  $\#E > d'$ . Let  $1 \leq m \leq s$  and  $D \in \text{Mat}_{s \times m}(\Lambda)$ . If there exists  $D_0 \in \text{GL}_s(\Lambda)$  such that  $D_0 = [*, D]$ , then there exist  $A \in \text{GL}_m(\Lambda)$  and  $B \in \text{GL}_s(E)$  such that*

$$BDA = \begin{bmatrix} 1 & & \\ & \ddots & \\ & & 1 \\ * & * & * \end{bmatrix} \in \text{Mat}_{s \times m}(\Lambda).$$

*Proof.* For each  $l \in \mathbb{Z}/d'$  and  $1 \leq j \leq m$ , let  $e_{l,j} \in \text{Mat}_{1 \times m}(\Lambda_l)$  be a row vector such that the  $j$ -th component is one and the other components are zero. Write  $D = (D_l)_l$  where  $D_l \in \text{Mat}_{s \times m}(\Lambda_l)$ . Since the rank of  $D_l$  is  $m$  for each  $l$ , there exists a matrix  $\tilde{A}_l \in \text{GL}_m(\Lambda_l)$  such that

$$D_l \tilde{A}_l = \begin{bmatrix} C_{l,1} \\ \vdots \\ C_{l,s} \end{bmatrix},$$

where  $C_{l,i} \in \text{Mat}_{1 \times m}(\Lambda_l)$ , and for each  $1 \leq j \leq m$  there exists an  $i$  such that  $C_{l,i} = e_{l,j}$ . An elementary pattern of  $D_l \tilde{A}_l$  is a choice of  $(i_{l1}, \dots, i_{lm}) \in \{1, \dots, s\}^m$  such that  $C_{l,i_{lk}} = e_{l,k}$  for each  $1 \leq k \leq m$ . We fix an elementary pattern  $(i_{l1}, \dots, i_{lm})$  of  $D_l \tilde{A}_l$  for each  $l$ . For



each matrix  $P \in \text{Mat}_{s \times m}(\Lambda_l)$  such that, for each  $1 \leq j \leq m$  there exists an  $i$  such that the  $i$ -th row of  $P$  is  $e_{l,j}$ , we define an elementary pattern of  $P$  in the same way. For a matrix in  $\text{Mat}_{s \times m}(\Lambda)$ , we define the procedures

- (1) left-multiplication by a matrix in  $\text{GL}_s(E)$ ,
- (2<sub>l</sub>) right-multiplication by a matrix in  $\text{GL}_m(\Lambda_l) \times \prod_{l' \neq l} \{1\}$ .

Set  $\tilde{A} := (\tilde{A}_l)_l \in \text{GL}_m(\Lambda)$  and  $C_i := (C_{l,i})_l \in \text{Mat}_{1 \times m}(\Lambda)$ . By using the above procedures, we want to transform  $D\tilde{A}$  to a matrix  $D' = (D'_l)_l$  such that, we can choose an elementary pattern of  $D'_l$  to  $(1, \dots, m)$  for each  $l$ .

Fix  $i' \neq i''$  and  $l_0$ . Let  $\tau = (i' \ i'')$  be the transposition of  $i'$  and  $i''$ . It is enough to show that, by using the procedures (1) and (2<sub>l</sub>), we can transform  $D\tilde{A}$  to a matrix  $D' = (D'_l)_l$  such that, we can choose an elementary pattern of  $D'_{l_0}$  to  $(\tau i_{l_0 1}, \dots, \tau i_{l_0 m})$  and an elementary pattern of  $D'_l$  to  $(i_{l1}, \dots, i_{lm})$  for each  $l \neq l_0$ .

First we assume that  $i' = i_{l_0 j'}$  and  $i'' = i_{l_0 j''}$  for some  $j' \neq j''$ . For  $c \in E^\times$ , we can exchange the  $i'$ -th row of  $D\tilde{A}$  for  $C_{i'} + cC_{i''}$  by the procedure (1). Since  $\#E > d'$ , we can take  $c$  such that, for each  $l \neq l_0$ , if  $i' = i_{lj}$  for some  $j$  then the  $j$ -th component of  $C_{l,i'} + cC_{l,i''}$  is non-zero. Then by the procedures (2<sub>l</sub>) for  $l \neq l_0$ , we can transform this matrix to a matrix  $D'' = (D''_l)_l$  such that, we can choose an elementary pattern of  $D''_l$  to  $(i_{l1}, \dots, i_{lm})$  for each  $l \neq l_0$ , the  $i$ -th row of  $D''_{l_0}$  is  $C_{l_0,i}$  for each  $i \neq i'$  and the  $i'$ -th row of  $D''_{l_0}$  is

$$(0, \dots, 0, \overset{j'}{\underset{\vee}{1}}, 0, \dots, 0, \overset{j''}{\underset{\vee}{c}}, 0, \dots, 0).$$

Therefore by the procedure (2<sub>l<sub>0</sub></sub>), we can transform  $D''$  to a matrix  $D'$  which has the desired properties. The case that  $i' \notin \{i_{l_0 1}, \dots, i_{l_0 m}\}$  and  $i'' = i_{l_0 j''}$  for some  $j''$  is proved in a similar way, and we omit the proof.  $\square$

**Lemma 5.9.** *Assume that  $\Gamma(F)$  is Zariski dense in  $\Gamma$  or  $\Lambda_l/F$  is a regular extension for each  $l$ . Assume also that  $\#E > d'$ . We take  $1 \leq m \leq s$  and  $D \in \text{Mat}_{s \times m}(\Lambda)$  such that  $[*, D] \in \text{GL}_s(\Lambda)$  for some  $* \in \text{Mat}_{s \times (s-m)}(\Lambda)$ . We set*

$$W := \{\mathbf{x} \in \text{Mat}_{1 \times s}(\Lambda') | \mathbf{x}D = 0\},$$

*and assume that  $\Gamma(F')W \subset W$ , where the elements of  $\Gamma(F')$  act on  $W$  by componentwise. Then there exists a matrix  $C \in \text{Mat}_{(s-m) \times s}(E)$  such that the rank of  $C$  is  $s - m$  and  $CD = 0$ .*

*Proof.* By Lemma 5.8, there exist matrices  $A \in \text{GL}_m(\Lambda)$  and  $B \in \text{GL}_s(E)$  such that

$$BDA = \begin{bmatrix} I_m \\ C_0 \end{bmatrix},$$

where  $I_m$  is the identity matrix of size  $m$  and  $C_0 \in \text{Mat}_{(s-m) \times m}(\Lambda)$ . We set

$$W_B := WB^{-1} = \{\mathbf{x} \in \text{Mat}_{1 \times s}(\Lambda') | \mathbf{x}BD = 0\} = \{\mathbf{x} \in \text{Mat}_{1 \times s}(\Lambda') | \mathbf{x} \begin{bmatrix} I_m \\ C_0 \end{bmatrix} = 0\}.$$

Then it is clear that  $W_B$  is also  $\Gamma(F')$ -stable. Thus, since each row of  $[-C_0 \ I_{s-m}]$  is an element of  $W_B$ , each row of  $[-\gamma C_0 \ I_{s-m}]$  is also an element of  $W_B$  for any  $\gamma \in \Gamma(F')$ .

This means that  $\gamma C_0 = C_0$  for each  $\gamma \in \Gamma(F')$ . Therefore  $C_0 \in \text{Mat}_{(s-m) \times m}(E)$  by Theorem 4.20. We set  $C := \begin{bmatrix} -C_0 & I_{s-m} \end{bmatrix} B$ . Then it is clear that this  $C$  has the desired properties.  $\square$

**Proposition 5.10.** *Assume that  $\Gamma(F)$  is Zariski dense in  $\Gamma$  or  $\Lambda_l/F$  is a regular extension for each  $l$ . Assume also that  $\#E > d'$ . For any  $N \in \mathcal{T}_M$  and  $\Gamma$ -subrepresentation  $U \subset \xi_M(N)$ , there exists a  $\varphi$ -submodule  $N' \subset N$  such that  $\xi_M(N') = N$ .*

*Proof.* We take  $\mathbf{u} \in \text{Mat}_{u \times 1}(U)$  an  $F$ -basis of  $U$  such that  $\bar{\mathbf{n}} := \begin{bmatrix} \mathbf{u} & * \end{bmatrix}^{\text{tr}}$  forms an  $F$ -basis of  $\xi_M(N)$ . By Lemma 5.2, we have  $\bar{\mathbf{n}} = H\mathbf{n}$  for some  $H = H(\Psi) \in \text{GL}_s(\Sigma)$ . We take a matrix  $D \in \text{Mat}_{s \times (s-u)}(\Sigma)$  such that  $H^{-1} = \begin{bmatrix} * & D \end{bmatrix}$ , and set  $W := \{\mathbf{x} \in \text{Mat}_{1 \times s}(\Lambda') \mid \mathbf{x}D = 0\}$ . Since  $I_s = HH^{-1} = \begin{bmatrix} * & HD \end{bmatrix}$ , the  $i$ -th row of  $H$  is an element of  $W$  for each  $i \leq u$ . These form a  $\Lambda'$ -basis of  $W$  because the coefficient ring  $\Lambda'$  is a finite product of fields. For each  $\gamma \in \Gamma(F')$ , we have  $\gamma\bar{\mathbf{n}} = (\gamma H)\mathbf{n} = (\gamma H)H^{-1}\bar{\mathbf{n}}$ . Since  $U$  is  $\Gamma$ -stable, the  $(i, j)$ -th component of  $(\gamma H)H^{-1} = \begin{bmatrix} * & (\gamma H)D \end{bmatrix}$  is zero for each  $i \leq u$  and  $j > u$ . Therefore,  $W$  is  $\Gamma(F')$ -stable. By Lemma 5.9, there exists a matrix  $C \in \text{Mat}_{u \times s}(E)$  such that the rank of  $C$  is  $u$  and  $CD = 0$ . Then we can take  $B \in \text{GL}_s(E)$  such that  $C$  forms the top rows of  $B$ . Let  $\begin{bmatrix} \mathbf{n}' & \mathbf{n}'' \end{bmatrix}^{\text{tr}} := B\mathbf{n}$  where  $\mathbf{n}' \in \text{Mat}_{u \times 1}(N)$ . Let

$$BH^{-1} = \begin{bmatrix} C \\ * \end{bmatrix} \begin{bmatrix} * & D \end{bmatrix} =: \begin{bmatrix} \Psi' & 0 \\ * & * \end{bmatrix},$$

where  $\Psi' \in \text{GL}_u(\Sigma)$ . Then we have

$$\begin{aligned} \varphi \begin{bmatrix} \mathbf{n}' \\ \mathbf{n}'' \end{bmatrix} &= \varphi(B\mathbf{n}) = \varphi(BH^{-1}H\mathbf{n}) = \sigma(BH^{-1})\varphi(H\mathbf{n}) = \sigma(BH^{-1})H\mathbf{n} \\ &= \sigma(BH^{-1})HB^{-1} \begin{bmatrix} \mathbf{n}' \\ \mathbf{n}'' \end{bmatrix} =: \begin{bmatrix} \Phi' & 0 \\ * & * \end{bmatrix} \begin{bmatrix} \mathbf{n}' \\ \mathbf{n}'' \end{bmatrix}, \end{aligned}$$

where  $\Phi' \in \text{GL}_u(E)$ . Hence  $N' := \langle \mathbf{n}' \rangle_E \subset N$  is a sub  $\varphi$ -module, and we have  $\varphi\mathbf{n}' = \Phi'\mathbf{n}'$ . Moreover, we have

$$\sigma \begin{bmatrix} \Psi' & 0 \\ * & * \end{bmatrix} = \sigma(BH^{-1}) = (\sigma(BH^{-1})HB^{-1})(BH^{-1}) = \begin{bmatrix} \Phi' & 0 \\ * & * \end{bmatrix} \begin{bmatrix} \Psi' & 0 \\ * & * \end{bmatrix} = \begin{bmatrix} \Phi'\Psi' & 0 \\ * & * \end{bmatrix}.$$

Therefore,  $\Psi'$  is a fundamental matrix for  $\Phi'$ . Since

$$\begin{bmatrix} \mathbf{n}' \\ \mathbf{n}'' \end{bmatrix} = B\mathbf{n} = BH^{-1}\bar{\mathbf{n}} = \begin{bmatrix} \Psi' & 0 \\ * & * \end{bmatrix} \begin{bmatrix} \mathbf{u} \\ * \end{bmatrix},$$

we have that  $\xi_M(N') = \langle (\Psi')^{-1}\mathbf{n}' \rangle_F = \langle \mathbf{u} \rangle_F = U$ .  $\square$

**Theorem 5.11.** *Assume that  $\Gamma(F)$  is Zariski dense in  $\Gamma$  or  $\Lambda_l/F$  is a regular extension for each  $l$ . Assume also that  $\#E > d'$ . Then the morphism of affine  $F$ -schemes  $\pi_M : \Gamma \rightarrow \Gamma_M$  is an isomorphism. Equivalently, the functor  $\xi_M : \mathcal{T}_M \rightarrow \mathbf{Rep}(\Gamma, F)$  is an equivalence of Tannakian categories.*

*Proof.* By Propositions 5.7 and 5.10,  $\pi_M$  is faithfully flat ([6], Proposition 2.21). On the other hand,  $\pi_M$  is a closed immersion by Theorem 5.6. Therefore  $\pi_M$  is an isomorphism.  $\square$

## 5.2 $v$ -adic case

In this subsection, we continue to use the notations of the previous subsection and consider the case that  $(F, E, L) = (\mathbb{F}_q(t)_v, K(t)_v, K^{\text{sep}}(t)_v)$ , where the notations are defined in Subsection 3.3.

The assumption that  $\Lambda_l/F$  is regular for each  $l$  is not true in general. For example, assume that  $r = 1$ ,  $v = t$  and  $\Phi \in K$  such that  $\Psi := \Phi^{1/(q-1)} \notin K(t)_v$ . Then  $\Psi$  is a fundamental matrix for  $\Phi$ , and it is clear that  $Z$  is not absolutely irreducible. Therefore the assumptions are not satisfied. However we expect that this assumption is true for “good” objects.

Hence we consider the other assumption. In the  $v$ -adic case,  $\Gamma(\mathbb{F}_q(t)_v)$  contains a Galois image. Since the Galois image is large enough, we can conclude that  $\Gamma(\mathbb{F}_q(t)_v)$  is Zariski dense in  $\Gamma$ .

**Lemma 5.12.** *Let  $G$  be an algebraic group over a field  $k$  and  $H$  a subgroup of  $G(k)$ . We set  $H^{\text{Zar}}$  the Zariski closure of  $H$  in  $G$  endowed with the reduced structure. Then  $H^{\text{Zar}}$  is a subgroup scheme of  $G$  and smooth.*

*Proof.* We denote by  $\bar{H}$  the Zariski closure of  $H$  in  $G(\bar{k})$ . By Lemma 4.17,  $\bar{H}$  is defined over  $k$ . Then it is clear that  $(H^{\text{Zar}})_{\bar{k}} = \bar{H}$ . Thus  $H^{\text{Zar}}$  is absolutely reduced.

To prove that  $H^{\text{Zar}}$  is a group scheme, it is enough to show that  $\bar{H}$  is a group. For any  $a \in G(\bar{k})$ , the map  $G(\bar{k}) \rightarrow G(\bar{k}); g \mapsto ag$  is a homeomorphism. Thus for any  $a \in H$ , we have  $a\bar{H} = \overline{aH} \subset \bar{H}$ . Thus for any  $b \in \bar{H}$ , we have  $Hb \subset \bar{H}$ . Therefore  $\bar{H}b = \overline{Hb} \subset \bar{H}$ . Hence we have  $\bar{H}\bar{H} \subset \bar{H}$ . Since the map  $G(\bar{k}) \rightarrow G(\bar{k}); g \mapsto g^{-1}$  is a homeomorphism, we have  $\bar{H}^{-1} = \overline{H^{-1}} = \bar{H}$ .  $\square$

**Lemma 5.13.** *Let  $G$  be a topological group and  $k$  be a topological field. Let  $\rho : G \rightarrow \text{GL}_r(k)$  be a continuous  $k$ -representation of  $G$ . We set  $\mathcal{C}_\rho$  the Tannakian subcategory of  $\mathbf{Rep}(G, k)$  generated by  $\rho$  and  $\Gamma_\rho \subset \text{GL}_r/k$  its Tannakian Galois group. Then  $\rho(G)$  is Zariski dense in  $\Gamma_\rho$ .*

Note that the Tannakian Galois group  $\Gamma_\rho$  may not be reduced.

*Proof.* We have an inclusion  $\rho(G)^{\text{Zar}} \subset \Gamma_\rho$  and  $\rho$  factors through a  $\rho(G)^{\text{Zar}}(k)$ :

$$\rho : G \rightarrow \rho(G)^{\text{Zar}}(k) \hookrightarrow \Gamma_\rho(k) \hookrightarrow \text{GL}_r(k).$$

Thus we have functors of Tannakian categories

$$\mathcal{C}_\rho \cong \mathbf{Rep}(\Gamma_\rho, k) \rightarrow \mathbf{Rep}(\rho(G)^{\text{Zar}}, k) \rightarrow \mathbf{Rep}(G, k).$$

We denote by  $\Gamma_{G,k}$  be the Tannakian Galois group of  $\mathbf{Rep}(G, k)$ . Then we have morphisms of algebraic groups which correspond to the above sequence:

$$\Gamma_{G,k} \rightarrow \rho(G)^{\text{Zar}} \hookrightarrow \Gamma_\rho.$$

Since  $\Gamma_{G,k} \rightarrow \Gamma_\rho$  is an epimorphism of algebraic groups, we have  $\rho(G)^{\text{Zar}}(\bar{k}) = \Gamma_\rho(\bar{k})$ .  $\square$

For any  $\tau \in G_K$ , since  $\sigma(\tau\Psi) = \tau(\sigma\Psi) = \tau(\Phi\Psi) = \Phi(\tau\Psi)$ , there exists a matrix  $A_\tau \in \text{GL}_r(\mathbb{F}_q(t)_v)$  such that  $\tau\Psi = \Psi A_\tau$ . Therefore we have  $\tau(\Sigma) = \Sigma$  and a map  $G_K \rightarrow \text{Aut}_\sigma(\Sigma/K(t)_v)$ . By Lemma 4.16, we have that  $A_\tau \in \Gamma(\mathbb{F}_q(t)_v)$  and  $A_\tau$  corresponds to the image of  $\tau$  in  $\text{Aut}_\sigma(\Sigma/K(t)_v)$  via the isomorphism  $\text{Aut}_\sigma(\Sigma/K(t)_v) \cong \Gamma(\mathbb{F}_q(t)_v)$ .

On the other hand, we can verify that the map

$$G_K \rightarrow \text{Aut}_\sigma(\Sigma/K(t)_v) \cong \Gamma(\mathbb{F}_q(t)_v) \hookrightarrow \Gamma_M(\mathbb{F}_q(t)_v) \hookrightarrow \text{GL}(V(M))$$

coincide with the natural representation  $G_K \rightarrow \text{GL}(V(M))$  defined in Subsection 3.3.

**Proposition 5.14.** *The image of  $G_K$  in  $\Gamma_M(\mathbb{F}_q(t)_v)$  is Zariski dense in  $\Gamma_M$ .*

*Proof.* Let  $\mathcal{C}_M$  be the Tannakian subcategory of  $\mathbf{Rep}(G_K, \mathbb{F}_q(t)_v)$  generated by  $V(M)$ . Then by Theorem 3.26, the categories  $\mathcal{T}_M$  and  $\mathcal{C}_M$  are equivalence. Therefore  $\Gamma_M$  is also a Tannakian Galois group of  $\mathcal{C}_M$ . Hence by Lemma 5.13, the image of  $G_K$  is Zariski dense in  $\Gamma_M$ .  $\square$

**Theorem 5.15.** *If  $(F, E, L) = (\mathbb{F}_q(t)_v, K(t)_v, K^{\text{sep}}(t)_v)$ , then the morphism  $\pi_M : \Gamma \rightarrow \Gamma_M$  is an isomorphism.*

*Proof.* By Proposition 5.14,  $\Gamma(\mathbb{F}_q(t)_v)$  is Zariski dense in  $\Gamma_M$ . In particular, it is Zariski dense in  $\Gamma$ . Therefore by Theorem 5.11,  $\pi_M$  is an isomorphism.  $\square$

**Proposition 5.16.** *Fix an index  $m \in \mathbb{Z}/d$  and take an element  $\tau \in G_K$  such that  $\tau|_{\mathbb{F}_{q^d}} = \sigma|_{\mathbb{F}_{q^d}}^m$ . Then the image of  $\tau$  in  $\Gamma(\mathbb{F}_q(t)_v)$  is contained in  $\Gamma_m(\mathbb{F}_q(t)_v)$ .*

*Proof.* Since  $\tau$  induces a  $K(t)_v$ -isomorphism  $K^{\text{sep}}((t - \lambda_{l+m})) \rightarrow K^{\text{sep}}((t - \lambda_l))$ , also induces a bijection  $Z_{l+m}(K^{\text{sep}}((t - \lambda_{l+m}))) \rightarrow Z_{l+m}(K^{\text{sep}}((t - \lambda_l)))$ . Let  $A_\tau \in \Gamma(\mathbb{F}_q(t)_v)$  be as above. Since  $\Psi A_\tau = \tau \Psi$ , we have  $\Psi_l A_\tau = \tau \Psi_{l+m} \in Z_{l+m}(K^{\text{sep}}((t - \lambda_l)))$ . Note that  $\Psi_l \in Z_l(K^{\text{sep}}((t - \lambda_l)))$  for each  $l$  by the definition of  $Z_l$ . Therefore by Theorem 4.11, we have  $A_\tau \in \Gamma_m(K^{\text{sep}}((t - \lambda_l))) \cap \Gamma(\mathbb{F}_q(t)_v) = \Gamma_m(\mathbb{F}_q(t)_v)$ .  $\square$

## 6 $v$ -adic criterion

In this section, we set  $K := \mathbb{F}_q(\theta)$  the rational function field over  $\mathbb{F}_q$  with one variable  $\theta$  independent of  $t$ . Let  $M$  be a finite-dimensional  $\varphi$ -module over  $K(t)_v$ ,  $\mathbf{m}$  a  $K(t)_v$ -basis of  $M$  and  $\Phi \in \text{Mat}_{r \times r}(K(t)_v)$  a matrix such that  $\varphi \mathbf{m} = \Phi \mathbf{m}$ .

**Definition 6.1.** A  $\varphi$ -module  $M$  is said to be a  $v$ -adic  $t$ -motive if  $\Phi \in \text{Mat}_{r \times r}(K[t]_v)$  and  $\det \Phi = c(t - \theta)^s$  for some  $c \in \bar{K}^\times$  and  $s \in \mathbb{N}$ .

Since  $t - \theta$  is invertible in  $K[t]_v$ ,  $v$ -adic  $t$ -motives are  $K^{\text{sep}}(t)_v$ -trivial by Theorem 3.26. Thus we can apply the results of the previous sections to  $v$ -adic  $t$ -motives.

**Remark 6.2.** Let  $k$  be a field of characteristic  $p > 0$  and  $\iota : \mathbb{F}_p[t] \rightarrow k$  a ring homomorphism. Anderson defined the notion of  $t$ -motives over  $k$  in [1]. This is a  $\varphi$ -module  $M$  over  $k[t]$  which satisfies the following conditions:

- $M$  is free of finite rank over  $k[t]$ .
- $(t - \iota(t))^N (M/k[t] \cdot \varphi M) = 0$  for some integer  $N > 0$ .
- $M$  is finitely generated over  $k_\sigma[\varphi]$ .

Here the  $\varphi$  action on  $k[t]$  is defined as before and  $k_\sigma[\varphi]$  is the subring of  $k[t]_\sigma[\varphi]$  generated by  $k$  and  $\varphi$ . Thus we have a functor from the category of  $t$ -motives over  $K$  (here we take  $\iota(t) = \theta$ ) to the category of  $v$ -adic  $t$ -motives by tensoring  $K(t)_v$ .

Let  $K_{v(\theta)}$  be the completion of  $K$  with respect to the place at  $v(\theta)$ ,  $K_d := K \cdot \mathbb{F}_{q^d}$  the composite field of  $K$  and  $\mathbb{F}_{q^d}$  in  $\bar{K}$ ,  $K_{\lambda_l} := K_{d,(\theta-\lambda_l)}$  the completion of  $K_d$  with respect to the place at  $(\theta - \lambda_l)$ ,  $\overline{K_{\lambda_l}}$  an algebraic closure of  $K_{\lambda_l}$ , and  $\mathbb{C}_{\lambda_l} := \widehat{\overline{K_{\lambda_l}}}$  the completion of  $\overline{K_{\lambda_l}}$  with respect to the canonical extension of  $(\theta - \lambda_l)$ . Let  $v_l$  be the valuation on  $\mathbb{C}_{\lambda_l}$  normalized by  $v_l(\theta - \lambda_l) = 1$ . For each  $l$ , we fix an embedding  $\bar{K}$  to  $\overline{K_{\lambda_l}}$  over  $K_d$ . Then for each  $f \in K^{\text{sep}}(t)_v = \prod_l K^{\text{sep}}((t - \lambda_l))$ , we can define  $f(\theta) \in \prod_l \mathbb{C}_{\lambda_l}$  by substituting  $\theta$  for  $t$  if it converge. We have the following conjecture, which is a  $v$ -adic analogue of Proposition 3.1.1 in [2]:

**Conjecture 6.3.** *Let  $\Phi \in \text{GL}_r(K(t)_v) \cap \text{Mat}_{r \times r}(K[t])$  and  $\psi \in \text{Mat}_{r \times 1}(K^{\text{sep}}[t]_v)$  be matrices such that  $\psi(\theta)$  converges,  $\sigma\psi = \Phi\psi$  and  $\det \Phi = c(t - \theta)^s$  for some  $c \in K^\times$  and  $s \in \mathbb{N}$ . Then, any linear relation of the components of  $\psi(\theta)$  over  $K_{v(\theta)}$  lifts to some linear relation of the components of  $\psi$  over  $K[t]_v$ . Precisely speaking, if there exists an element  $\rho \in \text{Mat}_{1 \times r}(K_{v(\theta)})$  such that  $\rho\psi(\theta) = 0$ , then there exists an element  $P \in \text{Mat}_{1 \times r}(K[t]_v)$  such that  $P\psi = 0$ ,  $P(\theta)$  converges and  $P(\theta) = \rho$ .*

Conjecture 6.3 is true if  $r = 1$  and we give a proof below. This proof is the same as the proof of the  $\infty$ -adic version for  $r = 1$  in [2].

If  $\rho = 0$ , then we can take  $P = 0$ . Therefore we may assume that  $\rho \neq 0$ . Since for some  $P \in K[t]_v$ , we have  $P(\theta) = \rho$ . Hence it is enough to show that, if  $\psi(\theta) = 0$ , then  $\psi = 0$ . Write  $\psi = (\sum_i a_{l,i}(t - \lambda_l)^i)_l$ . For any  $\nu \geq 0$ , the infinite sum  $\sum_i a_{l,i}((\theta - \lambda_l)^{q^{d\nu}})^i$  converge because  $\sum_i a_{l,i}(\theta - \lambda_l)^i$  converges and  $v_l((\theta - \lambda_l)^{q^{d\nu}}) \geq v_l(\theta - \lambda_l)$  for each  $l$ . Thus we have

$$\psi(\theta^{q^{d\nu}})^{q^d} = \left( \sum_i a_{l,i}((\theta - \lambda_l)^{q^{d\nu}})^i \right)_l^{q^d} = \left( \sum_i a_{l,i}^{q^d}(\theta^{q^{d(\nu+1)}} - \lambda_l)^i \right)_l = (\sigma^d \psi)(\theta^{q^{d(\nu+1)}}).$$

On the other hand, we have

$$\begin{aligned} (\sigma^d \psi)(\theta^{q^{d(\nu+1)}}) &= (\sigma^{d-1} \Phi)(\theta^{q^{d(\nu+1)}}) \times \dots \times (\sigma^0 \Phi)(\theta^{q^{d(\nu+1)}}) \times \psi(\theta^{q^{d(\nu+1)}}) \\ &= c^{q^{d-1} + \dots + q^0} (\theta^{q^{d(\nu+1)}} - \theta^{q^{d-1}})^s \dots (\theta^{q^{d(\nu+1)}} - \theta^{q^0})^s \psi(\theta^{q^{d(\nu+1)}}). \end{aligned}$$

By induction on  $\nu$ , we have  $\sum_i a_{l,i}((\theta - \lambda_l)^{q^{d\nu}})^i = 0$  for each  $l$ . Thus the formal series  $\sum_i a_{l,i}z^i$  has infinite zeros on the disk  $v_l(z) \geq v_l(\theta - \lambda_l)$ . Therefore  $a_{l,i} = 0$  for all  $l$  and  $i$ , and we conclude that  $\psi = 0$ .

Next, we calculate valuations of the coefficients of periods for some examples of  $t$ -motives. An element  $L_{\alpha,n}$  is an analogue of the  $n$ -th Carlitz polylogarithm, and an element  $\Omega_v$  is an analogue of the Carlitz period.

**Proposition 6.4.** *Let  $n \geq 1$  be an integer and  $\alpha \in (K^{\text{sep}})^\times$  an element such that  $v_l(\alpha) \geq 0$  for all  $l$ . Then there exists an element  $L_{\alpha,n} = L_{\alpha,n}(t) = (\sum_i a_{l,i}(t - \lambda_l)^i)_l \in K^{\text{sep}}[t]_v = \prod_l K^{\text{sep}}[[t - \lambda_l]]$  which satisfies the equation*

$$\sigma(L_{\alpha,n}) = \sigma(\alpha) + L_{\alpha,n}/(t - \theta)^n.$$

For any  $l \in \mathbb{Z}/d$ ,  $0 \leq m \leq d - 1$  and  $i \geq 0$ , we have

$$v_l(a_{l+m,i}) \geq -q^m \left( \frac{i}{q^d} + \frac{n}{q^d - 1} \right).$$

*Proof.* For an element  $L_{\alpha,n} = (\sum_i a_{l,i}(t - \lambda_l))_l \in \prod_l K^{\text{sep}}((t - \lambda_l))$ , we have an explicit descriptions

$$(t - \theta)^n \sigma(L_{\alpha,n}) = \left( \sum_i \left( \sum_{j=0}^n \binom{n}{j} (\lambda_l - \theta)^{n-j} a_{l-1,i-j}^q \right) (t - \lambda_l)^i \right)_l,$$

$$\sigma(\alpha)(t - \theta)^n = \left( \sum_{i=0}^n \binom{n}{i} (\lambda_l - \theta)^{n-i} \alpha^q (t - \lambda_l)^i \right)_l.$$

We set  $b_{l,i} := \sum_{j=1}^n \binom{n}{j} (\lambda_l - \theta)^{n-j} a_{l-1,i-j}^q - \binom{n}{i} (\lambda_l - \theta)^{n-i} \alpha^q$  and  $c_l := (\lambda_l - \theta)^n$ . Then the equation in Proposition is equivalent to the equations

$$a_{l+1,i} = c_{l+1} a_{l,i}^q + b_{l+1,i}$$

for all  $l \in \mathbb{Z}/d$  and  $i \in \mathbb{Z}$ . For  $i < 0$ , we can take  $a_{l,i} = 0$ . Fix  $i \geq 0$  and consider the system of polynomial equations

$$X_{l+1} = c_{l+1} X_l^q + b_{l+1,i} \quad (l \in \mathbb{Z}/d).$$

For  $2 \leq r \leq m$ , we set

$$\beta_{m,r,i} := b_{r,i}^{q^{m-r}} \prod_{s=r+1}^m c_s^{q^{m-s}} \quad \text{and} \quad \gamma_m := \prod_{s=2}^m c_s^{q^{m-s}}.$$

Then the above equations are equivalent to the equations

$$X_m = \gamma_m X_1^{q^{m-1}} + \sum_{r=2}^m \beta_{m,r,i} \quad (2 \leq m \leq d+1).$$

Since  $X_{d+1} = X_1$ , we can solve these equations in  $K^{\text{sep}}$ . This proved the existence part of this proposition.

Next we calculate the valuations of these solutions by induction on  $i$ . We set  $f_i(X_1) := \gamma_{d+1} X_1^{q^d} - X_1 + \sum_{r=2}^{d+1} \beta_{d+1,r,i}$ . Since  $a_{l,i} = 0$  for all  $i < 0$ , the inequality for the valuations in the statement of this proposition is true for  $i < 0$ . Fix  $i \geq 0$  and assume that the inequality in the statement of this proposition is true for integers lower than  $i$ . It is clear that  $v_1(\gamma_{d+1}) = v_1(c_1) = n$ . For  $2 \leq r \leq d$ , we have

$$\begin{aligned} v_1(\beta_{d+1,r,i}) &= n + q^{d+1-r} v_1(b_{r,i}) \\ &\geq n + q^{d+1-r} \min_{1 \leq j \leq n,i} \{v_1\left(\binom{n}{j}\right) + q v_1(a_{r-1,i-j}), v_1\left(\binom{n}{i}\right) + q v_1(\alpha)\} \\ &\geq n + q^{d+1-r} \min_{1 \leq j \leq n,i} \left\{ -q^{r-1} \left( \frac{i-j}{q^d} + \frac{n}{q^d-1} \right), 0 \right\} \\ &\geq n + q^{d+1-r} \left( -q^{r-1} \left( \frac{i-1}{q^d} + \frac{n}{q^d-1} \right) \right) \\ &= n - i + 1 - \frac{q^d n}{q^d - 1}. \end{aligned}$$

For  $r = d + 1$ , we have

$$\begin{aligned}
v_1(\beta_{d+1,d+1,i}) &= v_1(b_{1,i}) \\
&\geq \min_{1 \leq j \leq n,i} \{n - j + qv_1(a_{d,i-j}), n - i + qv_1(\alpha) + v_1\left(\binom{n}{i}\right)\} \\
&\geq \min_{1 \leq j \leq n,i} \left\{n - j - q^d \left(\frac{i-j}{q^d} + \frac{n}{q^d - 1}\right), 0\right\} \\
&\geq n - i - \frac{q^d n}{q^d - 1}.
\end{aligned}$$

Thus we conclude that  $v_1(\sum_{r=2}^{d+1} \beta_{d+1,r,i}) \geq n - i - q^d n / (q^d - 1)$ . By considering the Newton polygon of  $f_i$ , we have  $v_1(a_{1,i}) \geq -i/q^d - n/(q^d - 1)$  for any root  $a_{1,i}$  of  $f_i$ . For  $2 \leq r \leq m \leq d$ , we have

$$v_1(\beta_{m,r,i}) = q^{m-r} v_1(b_{r,i}) \geq q^{m-r} \left( -q^{r-1} \left( \frac{i-1}{q^d} + \frac{n}{q^d - 1} \right) \right) = -q^{m-1} \left( \frac{i-1}{q^d} + \frac{n}{q^d - 1} \right)$$

and

$$v_1(\gamma_m a_{1,i}^{q^{m-1}}) = q^{m-1} v_1(a_{1,i}) \geq -q^{m-1} \left( \frac{i}{q^d} + \frac{n}{q^d - 1} \right).$$

Thus we have

$$v_1(a_{m,i}) = v_1(\gamma_m a_{1,i}^{q^{m-1}} + \sum_{r=2}^m \beta_{m,r,i}) \geq -q^{m-1} \left( \frac{i}{q^d} + \frac{n}{q^d - 1} \right).$$

□

The next proposition is proved by similar arguments as Proposition 6.4.

**Proposition 6.5.** *There exists an element  $\Omega_v = \Omega_v(t) = (\sum_i a_{l,i}(t - \lambda_l))_l \in K^{\text{sep}}[t]_v^\times = \prod_l K^{\text{sep}}[[t - \lambda_l]]^\times$  which satisfies the equation*

$$(6.1) \quad \sigma(\Omega_v) = (t - \theta)\Omega_v.$$

For any  $l \in \mathbb{Z}/d$ ,  $0 \leq m \leq d - 1$  and  $i \geq 0$ , we have

$$v_l(a_{l+m,i}) = \frac{q^m}{q^{id}(q^d - 1)}.$$

By Propositions 6.4 and 6.5, the infinite sums  $L_{\alpha,n}(\theta)$  and  $\Omega_v(\theta)$  converge.

**Example 6.6.** We define the *Carlitz motive* to be the  $\varphi$ -module  $C$  whose underlying  $K(t)_v$ -vector space is  $K(t)_v$  and on which  $\varphi$  acts by

$$\varphi(f) = (t - \theta)\sigma(f)$$

for each  $f \in C$ . The equation (6.1) means that the element  $\Omega_v$  in Proposition 6.5 is a period of  $C$ . If we write  $\Omega_v = (\Omega_{v,l})_l = (\sum_i a_{l,i}(t - \lambda_l))_l$ , then  $[K_d(a_{l,0}, a_{l,1}, \dots) : K_d] = \infty$  by Proposition 6.5. Thus  $\Omega_{v,l}$  is transcendental over  $K(t)_v = K_d((t - \lambda_l))$ . Therefore we have that  $\text{tr.deg}_{K(t)_v} \Lambda_l = 1$  and  $\Gamma_C = \mathbb{G}_m$ .

## 7 Algebraic independence of formal polylogarithms

In this section, we prove the algebraic independence of certain “formal” polylogarithms. The proof of this theorem follows [5] and [10] closely. Let  $(F, E, L)$  be a  $\sigma$ -admissible triple and  $t, \theta \in E$  distinct elements. Let  $n, r$  be positive integers and  $\alpha_1, \dots, \alpha_r \in E$  fixed elements. Assume that  $(F^\times)_{\text{tor}} \neq F^\times$ , and there exist elements  $\Omega = (\Omega_l)_l \in L^\times$  and  $L_{\alpha_j, n} = (L_{\alpha_j, n, l})_l \in L$  for each  $j = 1, \dots, r$  such that  $\sigma(\Omega) = (t - \theta)\Omega$ ,  $\Omega_l$  is transcendental over  $E$  and  $\sigma(L_{\alpha_j, n}) = \sigma(\alpha_j) + L_{\alpha_j, n}/(t - \theta)^n$ . In the  $v$ -adic settings, such elements actually exist if  $\alpha_1, \dots, \alpha_r \in K^\times$  (cf. Section 6). We set

$$\Phi := \begin{bmatrix} (t - \theta)^n & & & \\ \sigma(\alpha_1)(t - \theta)^n & 1 & & \\ \vdots & & \ddots & \\ \sigma(\alpha_r)(t - \theta)^n & & & 1 \end{bmatrix} \text{ and } \Psi := \begin{bmatrix} \Omega^n & & & \\ \Omega^n L_{\alpha_1, n} & 1 & & \\ \vdots & & \ddots & \\ \Omega^n L_{\alpha_r, n} & & & 1 \end{bmatrix}.$$

Then we have  $\sigma\Psi = \Phi\Psi$ . Therefore, if  $M$  is the  $\varphi$ -module over  $E$  corresponding to  $\Phi$ , then  $M$  is  $L$ -trivial. This type of  $t$ -motive is considered in [5] and [10]. Note that in  $\infty$ -adic case,  $\Omega$  and  $L_{\alpha, n}$  are constructed explicitly, and  $L_{\alpha, n}(\theta)$  is the  $n$ -th Carlitz polylogarithm of  $\alpha$ . We define  $\Gamma, \Gamma_M, Z, \Lambda_l, \dots$  as in the previous sections for  $M, \Phi$  and  $\Psi$ . In particular, we have

$$\Lambda_l = E(\Omega_l^n, L_{\alpha, n, l}, \dots, L_{\alpha, n, l}).$$

Furthermore, we assume that,  $\Gamma(F)$  is Zariski dense in  $\Gamma$  or  $\Lambda_l/F$  is regular extension for each  $l$ . Thus the natural immersion  $\Gamma \rightarrow \Gamma_M$  is an isomorphism by Theorem 5.11.

For each  $F$ -algebra  $R$ , we set

$$G(R) := \left\{ \begin{bmatrix} * & 0 & \cdots & 0 \\ * & 1 & & \\ \vdots & & \ddots & \\ * & & & 1 \end{bmatrix} \in \text{GL}_{r+1}(R) \right\}.$$

Then  $G$  is an algebraic group over  $F$  and we have a natural inclusion  $\Gamma \subset G$ . Let  $X_0, \dots, X_r$  be the coordinates of  $G$  such that the first column of a general element of  $G$  “is”

$$\begin{bmatrix} X_0 \\ X_1 & 1 \\ \vdots & & \ddots \\ X_r & & & 1 \end{bmatrix}.$$

We have the exact sequence  $1 \rightarrow \mathbb{G}_a^r \rightarrow G \rightarrow \mathbb{G}_m \rightarrow 1$ , here  $\mathbb{G}_a^r$  is the subgroup scheme of  $G$  with coordinates  $(X_1, \dots, X_r)$  and  $\mathbb{G}_m$  is the quotient of  $G$  given by the projection  $(X_i) \mapsto X_0$ . Let  $C \in \Phi M_E^L$  be the one-dimensional  $\varphi$ -module such that  $\varphi(f) = (t - \theta)\sigma(f)$  for each  $f \in C = E$ . Then we have the following exact sequence:

$$0 \rightarrow C^{\otimes n} \rightarrow M \rightarrow \mathbf{1}^r \rightarrow 0.$$

Thus  $C^{\otimes n}$  is an object of  $\mathcal{T}_M$  and we have the canonical surjection  $\pi: \Gamma \cong \Gamma_M \rightarrow \Gamma_{C^{\otimes n}} \cong \mathbb{G}_m$ . We set  $V := \ker \pi$ . Then we have the commutative diagram

$$\begin{array}{ccccccc} 1 & \longrightarrow & V & \longrightarrow & \Gamma & \xrightarrow{\pi} & \mathbb{G}_m \longrightarrow 1 \\ & & \downarrow & & \downarrow & & \parallel \\ 1 & \longrightarrow & \mathbb{G}_a^r & \longrightarrow & G & \longrightarrow & \mathbb{G}_m \longrightarrow 1, \end{array}$$



where the rows are exact.

**Proposition 7.1.** *The subgroup  $V$  of  $\mathbb{G}_a^r$  is defined by linear forms in  $X_1, \dots, X_r$  with  $F$  coefficients.*

*Proof.* Let  $T \subset \Gamma_{\bar{F}}$  be a maximal torus and  $\bar{\pi}: \Gamma_{\bar{F}} \rightarrow \mathbb{G}_{m, \bar{F}}$  be the base extension of  $\pi$  to  $\bar{F}$ . Then we have  $\dim T = 1$  and  $\bar{\pi}|_T: T \rightarrow \mathbb{G}_{m, \bar{F}}$  is an isomorphism. Thus  $d\bar{\pi}$  is non-trivial and so is  $d\pi$ . Hence we have the following exact sequence:

$$0 \longrightarrow \text{Lie } V \longrightarrow \text{Lie } \Gamma \longrightarrow \text{Lie } \mathbb{G}_m \longrightarrow 0.$$

Since  $\Gamma$  and  $\mathbb{G}_m$  are smooth over  $F$ , we have the equalities  $\dim_F \text{Lie } \Gamma = \dim \Gamma$  and  $\dim_F \text{Lie } \mathbb{G}_m = 1$ . Thus we have the equality  $\dim_F \text{Lie } V = \dim V$ . Therefore  $V$  is smooth over  $F$ . Thus it is enough to show that the space  $V(\bar{F})$  is a linear space defined over  $F$ . Let

$$\mu = \begin{bmatrix} 1 & 0 \\ v & I_r \end{bmatrix} \in V(\bar{F}) \text{ and } \alpha \in \bar{F}^\times$$

be any elements. Since  $\Gamma(\bar{F}) \rightarrow \mathbb{G}_m(\bar{F})$  is surjective, there exists an element  $\gamma \in \Gamma(\bar{F})$  such that  $\pi(\gamma) = \alpha$ . Then we have

$$V(\bar{F}) \ni \gamma^{-1} \mu \gamma = \begin{bmatrix} 1 & 0 \\ \alpha v & I_r \end{bmatrix}.$$

Thus  $V(\bar{F})$  is a linear subspace of  $\mathbb{G}_a^r(\bar{F})$ . Since  $V$  is defined over  $F$ ,  $V$  is defined by linear forms in  $X_1, \dots, X_r$  with  $F$  coefficients.  $\square$

Since  $V$  is smooth and  $H^1(F, V) = 1$ , we have the exact sequence

$$1 \longrightarrow V(F) \longrightarrow \Gamma(F) \longrightarrow \mathbb{G}_m(F) \longrightarrow 1.$$

By the assumption on  $F$ , there exists an element  $b_0 \in F^\times \setminus (F^\times)_{\text{tor}}$ . By the above sequence, there exists an element

$$\gamma = \begin{bmatrix} b_0 & & & \\ b_1 & 1 & & \\ \vdots & & \ddots & \\ b_r & & & 1 \end{bmatrix} \in \Gamma(F).$$

We fix such  $b_0$  and  $\gamma$ . For each  $F$ -algebra  $R$  and  $a \in R^\times$ , we set

$$\gamma_a := \begin{bmatrix} a & & & \\ \frac{b_1}{b_0-1}(a-1) & 1 & & \\ \vdots & & \ddots & \\ \frac{b_r}{b_0-1}(a-1) & & & 1 \end{bmatrix}.$$

Then for each  $a, b \in R^\times$  and  $m \in \mathbb{Z}$ , we have  $\gamma_a \gamma_b = \gamma_{ab}$  and  $\gamma^m = \gamma_{b_0^m}$ . Hence we have  $\overline{\langle \gamma \rangle} = (R \mapsto \{\gamma_a | a \in R^\times\})$ , a line in  $\Gamma$ . We set  $\Gamma' := \overline{\langle V, \gamma \rangle} \subset \Gamma$  and  $s := r - \dim V$ . We claim that  $\Gamma' = \Gamma$ . Indeed, let

$$(7.1) \quad F_i = \sum_{j=1}^r c_{i,j} X_j \in F[X_1, \dots, X_r] \quad (i = 1, \dots, s)$$

be linear forms defining  $V$ . For each  $i$ , we set

$$G_i := (b_0 - 1)F_i(X_1, \dots, X_r) - F_i(b_1, \dots, b_r)(X_0 - 1) \in F[X_0, \dots, X_r].$$

Then we can verify that  $G_1, \dots, G_s$  define  $\Gamma'$  in  $\mathrm{GL}_{r+1}$  and  $\Gamma'$  is an algebraic group. Since  $V \subset \Gamma' \subset \Gamma$  and  $\Gamma' \rightarrow \mathbb{G}_m$  is surjective, we have  $\Gamma' = \Gamma$ . Thus we have the following proposition:

**Proposition 7.2.** *The algebraic group  $\Gamma$  is defined by the linear polynomials  $G_1, \dots, G_s$  in  $\mathrm{GL}_{r+1}/F$ .*

Since  $Z_{\bar{E}} \cong \Gamma_{\bar{E}}$  and  $Z$  is defined over  $E$ ,  $Z$  is defined by linear polynomials over  $E$ , and there exists an  $E$ -valued point

$$\xi = \begin{bmatrix} f_0 & & & & \\ f_1 & 1 & & & \\ \vdots & & \ddots & & \\ f_r & & & & 1 \end{bmatrix} \in Z(E).$$

We fix such  $\xi$ . Then we have  $Z = \xi \cdot \Gamma_E$ . Set  $f'_i := G_i(f_0, \dots, f_r)f_0^{-1} \in E$  and  $H_i := G_i(X_0, \dots, X_r) - X_0 f'_i \in E[X_0, \dots, X_r]$ . Then  $H_1, \dots, H_s$  are defining polynomials for  $Z$ . If we set  $g_i := \sum_{j=1}^r c_{i,j} b_j$ , then we have

$$H_i = (b_0 - 1) \sum_{j=1}^r c_{i,j} X_j + g_i - (g_i + f'_i) X_0.$$

Since  $\Psi_l \in Z(\Sigma_l)$  for each  $l$ , we have

$$(b_0 - 1) \sum_{j=1}^r c_{i,j} L_{\alpha_j, n, l} + g_i \Omega_l^{-n} - (g_i + f'_i) = 0$$

for each  $l$  and  $i$ . Set  $B := (c_{i,j})_{i,j} \in \mathrm{Mat}_{s \times r}(F)$ . By the definition of  $c_{i,j}$  (7.1), the rank of  $B$  is  $s = r - \dim V$ . Set

$$P = \begin{bmatrix} P_1 \\ \vdots \\ P_s \end{bmatrix} := \begin{bmatrix} (b_0 - 1)c_{1,1} & \dots & (b_0 - 1)c_{1,r} & g_1 & -(g_1 + f'_1) \\ \vdots & & \vdots & \vdots & \vdots \\ (b_0 - 1)c_{s,1} & \dots & (b_0 - 1)c_{s,r} & g_s & -(g_s + f'_s) \end{bmatrix} \in \mathrm{Mat}_{s \times (r+2)}(E),$$

the coefficients matrix of the above equations. Then the rank of  $P$  is also  $s$ . We interested in

$$N_l := \langle L_{\alpha_1, n, l}, \dots, L_{\alpha_r, n, l}, \Omega_l^{-n}, 1 \rangle_E \subset \Lambda_l.$$

This is the image of the  $E$ -linear map

$$\beta_l: E^{r+2} \rightarrow \Lambda_l; (x_1, \dots, x_{r+2}) \mapsto \sum_{j=1}^r x_j L_{\alpha_j, n, l} + x_{r+1} \Omega_l^{-n} + x_{r+2}.$$

Since  $P_i \in \ker \beta_l$  for each  $i$ , we have the inequality  $\dim_E \ker \beta_l \geq s$ . Thus we have  $\dim_E N_l \leq r + 2 - s = \dim V + 2 = \dim \Gamma + 1 = \mathrm{tr.deg}_E \Lambda_{l'} + 1$  for each  $l$  and  $l'$ . On the other hand, it is clear that  $\dim_E N_l \geq \mathrm{tr.deg}_E \Lambda_l = \mathrm{tr.deg}_E \Lambda_{l'}$ . Thus we have the following theorem:

**Theorem 7.3.** *For each  $l$  and  $l'$ , we have  $\text{tr.deg}_E \Lambda_{l'} \leq \dim_E N_l \leq \text{tr.deg}_E \Lambda_{l'} + 1$ .*

**Corollary 7.4.** *If  $L_{\alpha_1, n, l}, \dots, L_{\alpha_r, n, l}, 1$  are linearly independent over  $E$  for some  $l$ , then  $L_{\alpha_1, n, l'}, \dots, L_{\alpha_r, n, l'}$  are algebraically independent over  $E$  for each  $l'$ .*

*Proof.* Note that since  $\Lambda_{l'} = E(\Omega_{l'}^n, L_{\alpha_1, n, l'}, \dots, L_{\alpha_r, n, l'})$ , we have  $\text{tr.deg}_E \Lambda_{l'} \leq r + 1$ . By the assumption, we have  $r + 1 \leq \dim_E N_l \leq r + 2$ .

Assume that  $\dim_E N_l = r + 2$ . Then  $\text{tr.deg}_E \Lambda_{l'} < \dim_E N_l$ . By Theorem 7.3, we have  $\dim_E N_l = \text{tr.deg}_E \Lambda_{l'} + 1$ . Thus we have  $\text{tr.deg}_E \Lambda_{l'} = r + 1$  and  $\Omega_{l'}^n, L_{\alpha_1, n, l'}, \dots, L_{\alpha_r, n, l'}$  are algebraically independent over  $E$ .

On the other hand, assume that  $\dim_E N_l = r + 1$ . By the assumption, we can write  $\Omega_l^{-n}$  as a linear combination of  $L_{\alpha_1, n, l}, \dots, L_{\alpha_r, n, l}, 1$  over  $E$ . In particular, we have  $\Omega_l^n \in E(L_{\alpha_1, n, l}, \dots, L_{\alpha_r, n, l})$ . Letting  $\sigma$  act on this relation, we have

$$(t - \theta)^n \Omega_{l+1}^n \in \sigma(E) \left( \sigma(\alpha_1) + \frac{L_{\alpha_1, n, l+1}}{(t - \theta)^n}, \dots, \sigma(\alpha_r) + \frac{L_{\alpha_r, n, l+1}}{(t - \theta)^n} \right).$$

Thus for each  $l'$ , we have  $\Omega_{l'}^n \in E(L_{\alpha_1, n, l'}, \dots, L_{\alpha_r, n, l'})$  and  $\Lambda_{l'} = E(L_{\alpha_1, n, l'}, \dots, L_{\alpha_r, n, l'})$ . By Theorem 7.3, we have  $\text{tr.deg}_E \Lambda_{l'} \geq \dim_E N_l - 1 = r$ . Thus  $\text{tr.deg}_E \Lambda_{l'} = r$  and  $L_{\alpha_1, n, l'}, \dots, L_{\alpha_r, n, l'}$  are algebraically independent over  $E$ .  $\square$

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